Information and Optimal Trading Strategies with Dark Pools [*]

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June 2021

Abstract

We examine the competition between a transparent exchange organized as a limit order book and an opaque dark pool in the presence of asymmetric information. We show that price informativeness is generally lower when the dark pool and the exchange coexist. However, price informativeness might increase in the second trading period if there is no initial order migration to the dark pool. We also find that market liquidity increases initially for high fundamental volatility stocks and decreases for low liquidity stocks. Therefore, the impact of a dark pool that coexists with an exchange on market quality depends on stock market characteristics (fundamental volatility, liquidity, and adverse selection) and traders' characteristics (immediacy and information), and this allows us to derive new empirical and policy implications from our analysis.

Keywords: trading venues, dark liquidity, limit order book, price risk, adverse selection, double volume cap

JEL codes: G12, G14, G18

[*]The authors would like to thank Vladimir Asriyan, Michael Brolley, Jordi Caballé, Julio Crego, Martin Haferkorn, Javier Gil-Bazo, Chinmay Jain, Tomy Lee, Laurence Lescourret, Sophie Moinas, Roberto Pascual, José Penalva, Eghbal Rahimikia, Andreea Vaduva, Xavier Vives, and seminar participants at the Finance Theory Group Summer School 2017, ERMAS 2017, CEPR European Summer Symposium in Financial Markets 2017, Finance Forum 2017, CEPR-Imperial-Plato Conference, Swiss Society for Financial Market Research, 42nd Symposium of Economic Analysis, 2nd Catalan Economic Society Conference, 2019 Conference of the Society for the Advancement of Economic Theory, EARIE 2019, 3rd SAFE Market Microstructure Conference, Financial Management Association 2019 Annual Meeting, 10th Annual Financial Market Liquidity Conference, 37th International Conference of the French Finance Association, MIFID II three years later - IE Business School/CNMV Conference, Consob-ESMA-Bocconi Seminar Series, Toulouse School of Economics, ESSEC Business School, Universitat de les Illes Baleares, and ESADE Business School. The authors acknowledge financial support from the Government of Spain (AEI/FEDER, UE PID2019-105982GB-I00, FEDER/MICIU-AEI/PGC2018-098670, PR2015-00645), the Government of Catalonia (2017-SGR-770, 2017-SGR-640), Universitat Rovira i Virgili (2019PFR-URV-B2-53), Banc Sabadell, and the URL-La Caixa Foundation. Any errors are our own responsibility.

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Introduction

In today's financial markets, traders have access to competing trading venues with different levels of transparency. In addition to traditional exchanges (also known as lit markets), market participants can trade in dark pools (opaque trading venues). As of August 2020, trading volume in dark pools accounted for 12.31% market share of consolidated share volume in US markets and 6.32% of consolidated value traded in European markets.[1] The growing importance of dark pools as alternative trading venues, and the segmentation of the order flow into lit and dark venues has led regulators to focus on the impact of dark trading on market quality. Thus, in 2018, a regulatory reform in Europe, MiFID II, was implemented to offer greater protection for investors and foster transparency in financial markets. At the same time, in 2018 the U.S. Securities and Exchange Commission (SEC) also adopted amendments to Regulation ATS (Alternative Trading Systems) to enhance operational transparency and regulatory oversight.

Limit order books have replaced most of the traditional exchanges and compete for order flow with dark pools. Recent empirical work emphasizes the important role of information in the segmentation of the order flow and price discovery process when multiple trading venues compete (Ready, 2014; Comerton-Forde and Putniņš, 2015; Hatheway et al., 2017; Brogaard and Pan, 2021; Hendershott et al., 2020). The mixed results found in this literature and the regulators' concerns that dark trading harms price discovery suggest that there is a need for a new theoretical framework to understand price discovery and liquidity provision in markets where an exchange organized as a limit order book competes with a dark pool. Moreover, as information asymmetry plays a fundamental role in the price discovery process, a better understanding of the competition between a limit order book and a dark pool in the presence of asymmetric information is imperative.[2]

In this paper we examine how dark trading affects price discovery and market performance in a two-period model. Our model studies the role played by long-lived information on order choice selection by informed and uninformed traders, and the impact it has on market fragmentation. This is an important question since asymmetric information plays a crucial role in the decision to provide or demand liquidity in the limit order book or whether to submit an order to the dark pool. Moreover, the segmentation of the order flow alters the revelation of information in prices and therefore it may affect

[1] See Rosenblatt Securities, "Let there be light," and "Let there be light, European edition," August 2020.

[2] Bhattacharya and Saar (2020), Brolley and Malinova (2021), and Riccó et al. (2020) explain that the role of information on the order selection decision is not well understood theoretically, and it is methodologically challenging. Zhu (2014) also points out that the competition between an exchange that is organized as a limit order book and a dark pool is complex and difficult to solve analytically.

in time rational traders' optimal strategies. The current limit order book literature mainly focuses on the behavior of informed traders and simplifies the behavior of uninformed traders by introducing private values (Goettler, Parlour and Rajan, 2009) or time preferences (Rosu, 2020) to exogenously determine the trading strategies of uninformed traders. However, in practice, uninformed traders can infer information by observing the prices in the limit order book and use it when they choose their optimal trading strategy.[3] Therefore, in this paper we focus on how information held by traders impacts their order choices both in the lit and the dark markets.

In order to understand the effect of long-lived asymmetric information on the optimal traders' strategies we build a two-period trading model. In each trading period, a new trader arrives to the market and may trade one unit of a risky asset. There are two possible types of traders. On the one hand, there are rational traders who strategically choose whether or not to trade, and if they trade, they simultaneously select the venue and the type of order that maximize profits given their information. Rational traders can submit several order types: a market order or limit order to the exchange, or a dark pool order. In addition, rational traders may be informed if they know the liquidation value of the asset perfectly, or (privately) uninformed if they know only the distribution of the liquidation value of the asset conditional on public information. On the other hand, there are liquidity traders who participate in the market for liquidity reasons and submit market orders only to the exchange to ensure immediate execution.

Our model reflects the main characteristics of today's financial markets. The exchange is organized as a fully transparent limit order book that competes for order flow with a dark pool. Although many types of dark pools exist, our model makes two assumptions that capture their main features: (1) no pre-trade transparency because dark pools are completely opaque in the sense that they do not display available liquidity, which makes execution uncertain; and (2) dark pools do not determine prices and derive them from those prevailing in the exchange as the midpoint between the best bid and ask prices at any point in time.[4] A unique feature of our model with multiple venues is that, since the limit order book is transparent, an uninformed trader may learn about the liquidation value of the asset from the changes of prices in the limit order book. Another characteristic of our model with the dark pool is that we allow dark orders to be first routed to the dark pool and then re-routed to the exchange. As a result, we are able to study an important trade-off that traders face when submitting a dark order:

[3] We can interpret the first period of our model, as the time when an information shock occurs, and the second period as the period after this shock, when some of the private information is revealed through prices.

[4] This type of pricing is typical of dark pools owned by agency brokers or exchanges. Some dark pools offer other types of price improvements that are not equal to the midpoint (see, for example, Brolley, 2020).

they improve the execution price but face the risk that, in case of non-execution in the dark and the order returning to the exchange, the price in the limit order book has moved against them.

We compare traders' equilibrium strategies and market quality indicators in a model in which traders are restricted to trade only in the exchange (we refer to this setup as the single-venue market model) to one in which agents have access to a dark pool and an exchange (we refer to this setup as the two-venue market model). We show that in the first period traders choose to demand or provide liquidity in the exchange depending on market conditions (for example, informed traders demand/supply liquidity for high/low fundamental volatility stocks, while uninformed decide to supply liquidity or not to trade depending also on the degree of adverse selection they face) whenever the execution risk in the dark pool is high, and informed traders decide to migrate to the dark pool when the execution risk in the dark pool is low. We also show that in the first trading period in the two-venue market model, an informed trader finds dark orders more attractive than limit and market orders when the execution risk in the dark pool is sufficiently low. In contrast, an uninformed trader does not go to the dark pool in the first trading period since the price improvement is not sufficient to induce a change in trading venue. Nevertheless, the existence of a dark pool alongside the exchange may change the uninformed trader's optimal submission strategy from not trading to supplying liquidity in the exchange. This occurs when adverse selection in the exchange decreases because the informed trader migrates from the exchange to the dark pool. In such a case, there is order flow segmentation in the first trading period. In addition to the aforementioned factors, the optimal decisions in the second period depend on the information revealed in prices in the first trading period. Specifically, we find that both informed and uninformed traders submit dark orders if the execution risk in the dark pool is low enough and the price improvement is significant.

We show that dark trading has a negative impact on price informativeness for all stocks, in both periods, except when in the second period there is fragmentation of the order flow and informed traders choose to trade in the exchange while uninformed traders choose to trade in the dark pool. We show also that market liquidity increases initially for high fundamental volatility stocks and decreases for low liquidity stocks as traders migrating to the dark pool would have demanded and supplied liquidity in the exchange. Moreover, for both types of stocks, the expected profits of rational traders are higher in the two-venue market than in the single-venue market, except for uninformed traders if there is no migration to the dark pool initially, adverse selection is high and the uninformed traders' execution risk in the dark pool is large. Therefore, we find that the effects of dark trading may differ subsequently, although investors behave similarly in the first trading period. This is in contrast to other theoretical frameworks, and is due to the fact that traders learn from prices in the limit order book and optimally change their trading strategies accordingly.

Our work contributes to a growing body of theoretical research on the effects of competition between exchanges and dark pools. More specifically, we are the first to model the competition between an exchange organized as a limit order book and a dark pool in the presence of long-lived asymmetric information.[5] In a static set-up, Hendershott and Mendelson (2000) find that a crossing network (similar to a dark pool) that competes with a dealer market is characterized by positive liquidity externality and, at the same time, it generates a negative crowding externality, leading to ambiguous effects on market quality that depend on the insider's informational advantage. Degryse et al. (2009) show that the same positive and negative externalities remain in a dynamic setup and analyze how welfare and the order flow dynamics depend on the degree of market transparency. Ye and Zhu (2020) study how an informed trader splits his order between a dark pool and a dealer market, and show that he trades more aggressively in the dark pool than on the exchange.[6]

Our paper is more closely related to Zhu (2014), Buti et al. (2017) and Brolley (2020). Like in Zhu (2014), we examine the role of asymmetric information in competing trading venues. However, we model the competition of a dark pool with a limit order book; therefore, in our framework, traders can both demand liquidity (as in Zhu, 2014) and supply liquidity to the exchange. Moreover, we propose a two-period model, which allows for the first time to examine how information is gradually incorporated in the limit order book, and how traders' strategies reflect this change. Interestingly, under some parameter configurations, we find the same result as Zhu (2014) that dark pools improve price informativeness (when in the second trading period the informed stays in the exchange and the uninformed trades in the dark pool). In contrast, we show that when market conditions are such that the informed trader migrates to the dark pool and the uninformed stays in the exchange, the existence of the dark pool harms price informativeness.[7]

Buti et al. (2017) and Brolley (2020) examine the competition between a fully transparent limit

[5]Glosten (1994), Chakravarty and Holden (1995), Seppi (1997), Kaniel and Liu (2006) emphasize the role of asymmetric information in the order submission strategies' choice in a single trading venue. Parlour (1998), Foucault (1999), Parlour and Seppi (2003), Foucault et al. (2005), Goettler et al. (2009), Rosu (2009), Brolley and Malinova (2021), and Riccó et al. (2020) study the optimal choice of order type in dynamic models.

[6]Our research is also related to two other broader strands of the literature: competition between multiple trading venues (see Gomber et al., 2016 for a review of the literature) and transparency (Biais, 1993; Madhavan 1995; Frutos and Manzano, 2002, 2005; Dumitrescu, 2010; Boulatov and George, 2013, among others).

[7]Ye (2011) finds that adding a dark pool alongside a dealer market always reduces price informativeness if the uninformed is restricted to trading in the exchange.

order book and a dark pool. In a symmetric information setup with private values, Buti et al. (2017) show that the introduction of a dark pool that competes with an illiquid limit order book is, on average, associated with trade creation, wider spreads, lower depth, and welfare deterioration. To complement their work, we introduce asymmetric information in a common value setup and find the same results as in Buti et al. (2017) for low fundamental volatility stocks (when information is of low value for an informed trader) in the first trading period. However, since traders learn from prices in the second trading period, our market quality results differ fundamentally. In a model with asymmetric information, Brolley (2020) shows that the impact of dark trading on market quality depends on the relative price improvement of dark orders over limit orders. In contrast to Brolley (2020), we develop a model in which the dark pool reference price is the midpoint of the exchange and where, in general, the market conditions in the two trading periods differ. We characterize how the effects of a dark pool that competes with an exchange on market quality depend on the market quality indicator, trading period, and stock and trader characteristics. The differences in both trading periods emerge because in our two-period trading model with long-lived information, an uninformed trader uses the prices in the limit order book to extract information about the common value of the asset.

The fact that market performance indicators depend fundamentally on stock and trader characteristics is novel and helps us reconcile the mixed results reported in the empirical literature on the effects of dark pools on the market performance of the exchange. In terms of price informativeness, our results that price informativeness may decrease are consistent with the empirical results reported by Hendershott and Jones (2005) and Comerton-Forde and Putniņš (2015) (for high levels of dark trading; i.e., above 10%), while our results that price informativeness may increase (in the second trading period) are related to the empirical results reported by Ready (2014) and Comerton-Forde and Putniņš (2015) (for low levels of dark trading). We also show that in the two-venue market in which a low fundamental volatility stock is traded there is a negative effect on liquidity in the first trading period (which is consistent with the results of Degryse et al., 2015; and Hatheway et al., 2017), while when a high fundamental volatility stock is traded there is a positive effect on liquidity (Buti et al., 2011; Ready, 2014). In addition, our model also provides new empirical implications regarding changes in market quality, both in the time-series and the cross-section, emphasizing the role of stock and trader characteristics in the decision on whether to supply or demand liquidity in the exchange.

Moreover, we provide implications for the current policy debate regarding the regulation of dark pools as our results point out that regulators should be cautious in choosing a "one-size-fits-all" regulatory policy, given potential unintended consequences. This is the case, for example, for the MiFID II rule known as the double volume cap (DVC) that limits the volume of dark trading in one venue at 4% and the entire market at 8%. Consistent with our model implications regarding the effectiveness of dark trading restrictions, European Securities and Markets Authority (ESMA) recently decided to remove the volume cap that limits dark trading in a single venue at 4%.

1 Model

We consider a market in which a single risky asset is traded. The liquidation value of the asset, $\widetilde{V}$, may take two values, $V \in \{V^H, V^L\}$, with equal probabilities. We denote the unconditional mean of $\widetilde{V}$ by μ and $\sigma > 0$ represents the fundamental volatility (i.e., standard deviation). The asset may be traded in two venues: an exchange, organized as a limit order book (LOB), or a dark pool (DP).[8]

At the beginning of the game, potential participants in the exchange have access to an electronic book that provides an anonymous list of previously entered limit orders. Specifically, the initial LOB has at least three prices on the ask and bid sides of the book: A_1^1, A_1^2, A_1^3, and B_1^1, B_1^2, B_1^3, respectively, such that $V^L \leq B_1^3 < B_1^2 < B_1^1 < \mu < A_1^1 < A_1^2 < A_1^3 \leq V^H$. In addition, prices are placed on a grid and the following relationships hold:

$$
\begin{aligned}
A_1^1 &= \mu + k_1\tau,\ A_1^2 = \mu + k_2\tau,\ A_1^3 = \mu + k_3\tau,\ V^H = \mu + \kappa\tau, \\
B_1^1 &= \mu - k_1\tau,\ B_1^2 = \mu - k_2\tau,\ B_1^3 = \mu - k_3\tau,\ V^L = \mu - \kappa\tau,
\end{aligned}
$$

with $1 \leq k_1 < k_2 < k_3 \leq \kappa$, where k_1, k_2, and k_3 are natural numbers, and τ is the tick size (i.e., the minimum price change that traders are allowed to quote over the existing price). Note that the volatility of the asset satisfies $\sigma = \kappa\tau$, with κ being a real number.

The way the price grid works in our model allows us to start with a full or an almost empty book depending on the parametrization.[9] Thus, for low values of k_1 the book has orders with prices that are close to the midpoint – the mean of the liquidation value of the asset – and therefore, the book is similar to a full book. However, for very high values of k_1 close to κ, traders will behave as if the book

[8]For ease of reading, refer to Appendix A which contains a summary table with the notation used.

[9]Some LOB models assume that the book starts empty (Seppi, 1997; Buti and Rindi, 2013; Buti et al., 2017; Riccó et al., 2020), that is, the only standing limit orders in the initial book are those at "extreme" prices coming from a trading crowd. As Riccó et al. (2020) point out this is a simplification given that, in practice, daily opening limit order books include uncancelled orders from the previous day and new limit orders from opening auctions. In these models, in the first trading period, if an investor wants to trade, he always selects a LO. By contrast, in an initial non-empty LOB (as in Parlour, 1998 and Foucault, 1999) the trader can select any type of order.

was empty.[10] We can interpret $1/k_1$ as a measure of stock liquidity, so the market is very liquid when $k_1 = 1$. For simplicity, we assume that the initial depth of the LOB at each bid and ask price is equal to 1, and that the LOB follows price and time priority rules. The LOB is fully transparent (i.e., all of the information in the LOB is available to all market participants at any point in time). Traders can submit market orders or limit orders to the LOB. There are no transaction costs or trading fees.

The DP is completely opaque in the sense that an order submitted to the DP is not observable to anyone besides the trader who submitted it. Orders in the DP are executed continuously. In our setting, this means that the DP crosses orders at each trading round, whenever possible. If a trader submits an order to the DP and it is executed at t, then the execution price is equal to the midpoint of the exchange at t: $(A_t^1 + B_t^1)/2$, where A_t^1 and B_t^1 denote the best ask and bid prices at the beginning of trading period t. If the order is not executed in the DP at t, then the trader can cancel it or keep it. If the trader keeps the order, then it returns to the exchange at $t+1$.[11] This feature allows us to model an important trade-off that traders face when submitting a dark order: they obtain a price improvement, but face the risk of non-execution in the dark and the possibility that when their orders return to the market the price has moved against them.

Following Degryse et al. (2009), and to keep the tractability of our model with asymmetric information, we assume that the liquidity supply in the DP at $t = 1$ is exogenous, however, the probability of execution in the DP becomes endogenous at $t = 2$. Thus, similarly to the LOB, the initial liquidity in the DP is provided by traders outside the model. In practice, the DP order flow is substantially fragmented partly due to the emergence of order routing and splitting algorithms (Gresse, 2017). Thus, DP orders come from multiple and diverse sources, including the order flow broker-dealers, institutional traders, liquidity providers and proprietary flow. Moreover, the orders that traders might send to the DP in our model are small in relation to the aggregate liquidity in the DP.

Concerning traders, we assume that all traders are risk neutral and may trade one unit of the asset (as in Glosten and Milgrom, 1985; Foucault, 1999; and Riccó et al., 2020). There are two possible types of traders: rational and liquidity traders.[12] Rational traders choose an order submission strategy that

[10] We do not model a trading crowd willing to provide liquidity at the highest possible prices (as in Seppi, 1997 and Parlour, 1998). In their setup, this assumption prevents traders from bidding prices that are too distant from the inside spread. In our framework, since we assume that there are at least three prices previously populated with orders in the LOB and order size is 1, traders will not get to trade against this crowd.

[11] In practice, a trader can implement the decision to cancel or keep an order when it is not executed in the DP using smart order routing. Orders are automatically filled while sweeping for liquidity at the available trading venues.

[12] Rational traders may be institutional traders that monitor liquidity using algorithms and strategically submit orders (see Malinova et al. 2018), while liquidity traders can be understood as retail investors that typically do not have this technology and trade due to liquidity needs.

maximizes their expected profits conditional on their information sets which include the information provided by the LOB. Rational traders simultaneously select whether or not to trade (NT), and if they trade, they choose the trading venue (exchange or DP), and the order type in the exchange (market order, MO, or limit order, LO). DO represents the order type in the DP. Consequently, the set of strategies available to a rational trader (both informed and uninformed) is

$$\mathbb{O}_D = \{MO,\ LO,\ DO,\ NT\}, \tag{1}$$

where a B in front of an order type denotes a buy order and a S a sell order. $\Pi^R_{\mathcal{O},t}$ represents the profits of a particular order, where the superscript R denotes that the order comes from a rational trader ($R = I, U$, where I and U indicate informed and uninformed traders, respectively); subscript $\mathcal{O}$ is the order type $\mathcal{O} \in \mathbb{O}_D$ defined in (1), and the subscript t is the order submission date.

The sequence of events is illustrated in Figure 1.

$t = 0$	$t = 1$	$t = 2$	$t = 3$
Nature draws V $V \in \{V^H, V^L\}$	• A trader arrives • If he is rational, he chooses his optimal strategy based on his information set • If he is a liquidity trader, he trades according to his liquidity needs	• A new trader arrives • If he is rational, he chooses his optimal strategy based on his information set • If he is a liquidity trader, he trades according to his liquidity needs • If at $t = 1$ a DO was submitted and not executed, the order returns to the exchange	• Liquidation value is made public • If at $t = 2$ a DO was submitted and not executed, the order returns to the exchange • The trading game is over

Figure 1: Timeline of the trading game when traders have access to the DP.

Figure 2 illustrates the tree of events for the first trading period.[13] A rational trader arrives at the market with probability $\lambda > 0$ and a liquidity trader arrives with probability $1 - \lambda > 0$. Rational traders may be either (privately) informed if they have perfect information about the liquidation value of the asset (with probability $\pi > 0$), or (privately) uninformed if they know only the distribution of the liquidation value of the asset (with probability $1 - \pi$).[14] We use $PIN \equiv \lambda\pi$, the probability

[13] We can draw a similar tree of events for the second trading period.

[14] Uninformed traders are those that only use public information while informed traders use both public and private information. For example, an uninformed trader could be a fund manager that rebalances his portfolio for non-informational reasons (see Han et al., 2016), while an informed trader may be a fund manager who uses his connections to acquire information (see Coval and Moskowitz, 2001; Cohen et al. 2008 among others).

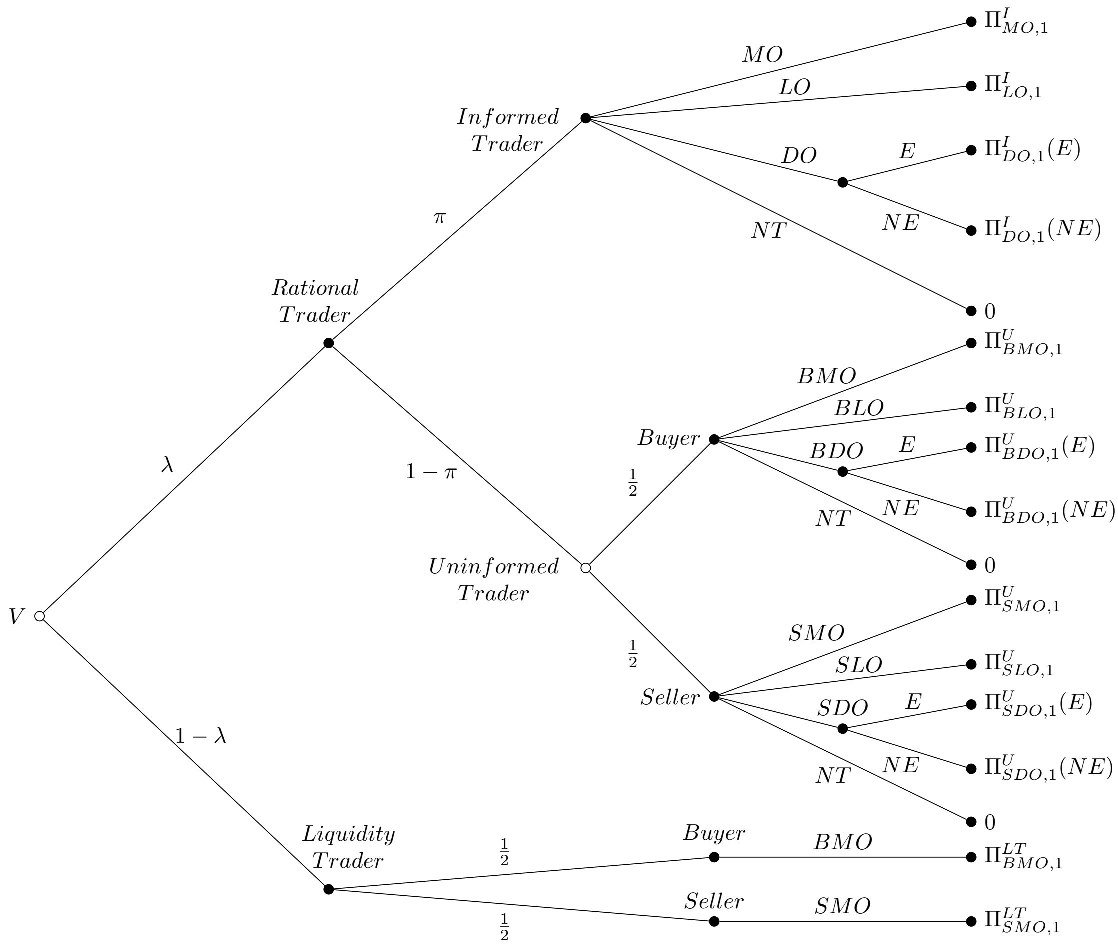

Figure 2: Tree of events of the first trading period.

of informed trading as a measure of information asymmetry, following Easley and O'Hara (1987) and Easley et al. (1996). An informed trader buys when observing $V = V^H$ (denoted by IH), and sells when observing $V = V^L$ (denoted by IL). An uninformed trader is a buyer (denoted by UB) with probability $\frac{1}{2}$ or a seller (denoted by US) with probability $\frac{1}{2}$. A liquidity trader buys with probability $\frac{1}{2}$ or sells with probability $\frac{1}{2}$ for liquidity or hedging needs.

Uninformed and liquidity traders both have an intrinsic motive to trade (although their motives for purchasing or selling the asset are not explicitly modelled). However, they differ in their immediacy needs: liquidity traders are impatient and only use market orders, while uninformed traders are patient and rationally choose whether to trade, the order type and the venue. Moreover, uninformed traders can learn from the changes in the LOB, so their orders may change from one period to another.[15]

[15] Liquidity traders are modelled to foster the trading of uninformed traders.

The final nodes of the tree include the profits for each of the trading options at $t = 1$. At the end of the first period, the possible state of the LOB (possible best prices of the LOB) can be: $(A_1^2, B_1^1), (A_1^1, B_1^2), (A_1^1, B_1^1), (A_1^1, B_1^1 + \tau), (A_1^1 - \tau, B_1^1)$.

The structure of the model and distributions of random variables are common knowledge.

For each possible order type, we next examine its characteristics and the associated expected profits for a rational buyer (the sell order profits are analogous). Internet Appendix I derives in detail the expected profits of all traders at all times and for all possible states of the LOB.

Market order (MO): It is executed immediately at the given best available ask/bid prices. The expected profits of a BMO at date t are

$$\mathbb{E}\left(\Pi_{BMO,t}^R | I_t\right) = \mathbb{E}\left(\widetilde{V} | I_t\right) - A_t^1,$$

where I_t is information set at each date t.

Limit order (LO): A LO that improves the current market price may be executed in the next period if a MO of the opposite sign hits the LOB. Thus, LOs provide better prices than MOs do but have execution risk. When a trader chooses a LO, this order always improves the current price by one tick because: (i) it is never optimal for the trader to improve the price by more than one tick since it reduces his profits; (ii) it is never optimal for the trader to submit a non-improving LO since the order is not executed (due to time priority, the order goes to the end of the queue), and obtains zero profits. Given that we also assume a discount factor, $\delta \in (0, 1]$, which is common across traders and periods, the expected profits of a BLO at date t are

$$\mathbb{E}\left(\Pi_{BLO,t}^R | I_t\right) = \delta p_{BLO,t}^R\left(I_t\right)\left(\mathbb{E}\left(\widetilde{V} | I_t\right) - \left(B_t^1 + \tau\right)\right),$$

where $p_{BLO,t}^R$ is the probability of execution of a BLO submitted by a rational trader, R, at time t.

Dark order (DO): With probability θ_t^R, an order submitted by a rational trader, R, to the DP at time t is executed, and with probability $(1 - \theta_t^R)$ it is not executed. Since no new trader arrives in the market at $t = 3$ an order that returns to the exchange from the DP at the end $t = 2$ will be either a MO (we call this dark order $BDO - MO$) or NT (we call this order

$BDO-NT$).[16] We denominate the DO as the best of the two: $BDO-MO$ and $BDO-NT$ (for each type of trader).[17] Note that a DO does not change the state of the LOB, and to model the reporting delay of DP trades, we consider that the DP does not report trades until the end of the trading game. Therefore, the expected profits of a BDO submitted at time t are

$$\begin{aligned}\mathbb{E}\left(\Pi^R_{BDO,t}|I_t\right) &= \max\left\{\mathbb{E}\left(\Pi^R_{BDO-MO,t}|I_t\right), \mathbb{E}\left(\Pi^R_{BDO-NT,t}|I_t\right)\right\} \\ &= \theta^R_t\left(\mathbb{E}\left(\widetilde{V}|I_t\right) - \frac{A^1_t + B^1_t}{2}\right) + (1-\theta^R_t)\delta\max\left\{\mathbb{E}\left(\Pi^R_{BMO,t+1}|I_t\right), 0\right\}.\end{aligned}$$

No trade (NT): A trader who refrains from trading at t obtains zero profits:

$$\mathbb{E}\left(\Pi^R_{NT,t}|I_t\right) = 0.$$

In case of equal profits, we assume that a MO dominates both a LO and a DO, and a LO dominates a DO. If the expected profits of a MO are null, then a rational trader refrains from trading.

We can represent our model by a two-period game of incomplete information, and we therefore use the Perfect Bayesian Equilibrium (PBE) concept. In the following, we focus on a symmetric PBE in pure strategies, hereafter, equilibrium.[18]

2 Equilibrium in the single-venue market model

We first consider the single-venue market - where traders can only trade in the exchange. Hence, the set of strategies available to a rational trader is $\mathbb{O}_D \backslash \{DO\}$, that is, a MO, a LO, and NT.

We solve the game backwards. Since the buy and sell sides are separable and symmetric in this model, we focus for exposition on the buy side. The expected profits for the rational traders at $t=2$ are summarized in Appendix B, Tables B.1 and B.2, while Tables B.4 and B.5 display the expected profits for these traders at $t=1$. The following lemma presents the informed and uninformed traders' optimal choices at $t=2$ and $t=1$.

[16] Since the probability of execution of a LO at $t=3$ is 0, an order will never return to the market as a LO.

[17] As we show in the Internet Appendix I, when an informed trader chooses a DO at $t=1$ and the order is not executed, it is optimal for the informed trader to choose a MO when the order returns to the exchange at the end of the second trading period (i.e., $DO-MO$). In contrast, when an uninformed trader chooses a DO at $t=1$ and the order is not executed, it is optimal for the uninformed to cancel it at the end of the second trading period (i.e., $DO-NT$). However, at $t=2$ both types of traders are indifferent between $DO-MO$ and $DO-NT$ since at $t=3$ the liquidation value is revealed and the profits of both strategies are zero.

[18] A symmetric equilibrium refers to a situation in which buyers and sellers with the same information (i.e., informed or uninformed) choose the same order type (except the direction of trade).

Lemma 1 *In equilibrium, the following results hold:*

At $t = 2$, an informed trader always submits a MO, while an uninformed trader may submit either a MO or NT, but never chooses a LO.

At $t = 1$, an informed trader may submit either a MO or a LO, but never chooses NT, while an uniformed trader may submit either a LO or NT, but never chooses a MO.

Thus, the candidate strategy profiles at $t = 1$ that can be sustained as a symmetric PBE are:

$$\mathcal{E}_1^{ND} : (BMO,\ SMO,\ BLO,\ SLO), \qquad \mathcal{E}_2^{ND} : (BMO,\ SMO,\ NT,\ NT),$$

$$\mathcal{E}_3^{ND} : (BLO,\ SLO,\ BLO,\ SLO), \qquad \mathcal{E}_4^{ND} : (BLO,\ SLO,\ NT,\ NT),$$

where the first two components correspond to the strategies of informed traders at $t = 1$ (IH and IL, respectively) and the last two components correspond to the strategies of uninformed traders at $t = 1$ (UB and US, respectively).[19]

The next proposition describes the symmetric PBE of the trading game in the single-venue market.

Proposition 1 *In the single-venue market:*

Case A. *If $k_1 > 1$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BLO, SLO, BLO, SLO) & \text{if } \ \sigma < \kappa^I_{MO-LO}\tau \text{ and } PIN < \psi^U_{LO-NT}, \\ (BLO, SLO, NT, NT) & \text{if } \ \sigma < \kappa^I_{MO-LO}\tau \text{ and } PIN \geq \psi^U_{LO-NT}, \\ (BMO, SMO, BLO, SLO) & \text{if } \ \kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN < \psi^U_{LO-NT}, \\ (BMO, SMO, NT, NT) & \text{if } \ \kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN \geq \psi^U_{LO-NT}, \end{cases}$$

where $\kappa^I_{MO-LO} \equiv (k_1 - 1) + 2\frac{\delta(k_1-1)(1-\lambda)+1}{2-\delta(1-\lambda)}$, $PIN \equiv \lambda\pi$, *and* $\psi^U_{LO-NT} \equiv \frac{(1-\lambda)(k_1-1)\tau}{\sigma-(k_1-1)\tau}$.

Case B. *If $k_1 = 1$ (the asset is very liquid), then the optimal strategy profile at $t = 1$ is (BMO, SMO, NT, NT).*

For Cases A and B, the optimal strategy of an informed trader at $t = 2$ is to choose a MO for all possible states of the LOB, while an uninformed trader chooses either a MO or NT depending on his beliefs and market conditions.

[19] In what it follows the superscript ND indicates that there is no access to the DP, while D indicates that there is access.

Remark 1 *Notice that* κ^I_{MO-LO} *denotes the minimum value of* κ *such that at* $t = 1$ *an informed trader chooses a* MO *instead of a* LO*, while* ψ^U_{LO-NT} *represents the minimum value of* PIN *such that at* $t = 1$*, an uninformed trader chooses* NT *instead of a* LO*.*

In the second trading period, according to Lemma 1, Proposition 1 shows that an informed trader submits a MO for all states of the LOB. An uninformed trader chooses NT, except if the state of the LOB conveys information about the fundamental value of the asset and he strongly believes that a MO or LO of the same direction as his order was submitted by an informed trader at $t = 1$. In this case, the uninformed trader submits a MO.

In the first trading period, Proposition 1 indicates that when the fundamental asset volatility is sufficiently low (i.e., $\sigma < \kappa^I_{MO-LO}\tau$), it is optimal for an informed trader to supply liquidity (i.e., to place a LO), while the decision of the uniformed trader depends on the severity of the adverse selection problem. Therefore, there are two possible optimal strategy profiles when the asset has low volatility: (BLO, SLO, BLO, SLO) and (BLO, SLO, NT, NT). The optimal strategy profile (BLO, SLO, BLO, SLO) occurs in a market with low adverse selection risk (either because the asset's volatility is extremely low, or both the asset's volatility and the PIN are low at the same time). When the adverse selection problem is sufficiently high (because the asset's volatility is not low, and the PIN is high enough) the optimal strategy profile is (BLO, SLO, NT, NT). In particular, when the probability of informed trading is low, uninformed traders realize that by placing a LO at the exchange in the first trading period, they are very unlikely to end up trading with informed traders. Combining the fundamental volatility and information asymmetry dimensions, we call these stocks *"Low-Low"* (i.e., low fundamental volatility– low PIN), and *"Low-High"*, respectively. In the subsequent analysis, we sometimes consider only one of these dimensions in isolation, such as low/high fundamental volatility stocks or low/high PIN.

By contrast, when the fundamental asset volatility is sufficiently high (i.e., $\sigma \geq \kappa^I_{MO-LO}\tau$), it is optimal for the informed trader to demand liquidity (i.e., to place a MO) in the first trading period. Note that the informational advantage of an informed trader increases with the volatility of the asset (σ). Thus, when the asset's volatility is sufficiently high, an informed trader prefers immediate execution (MO). When σ is not high enough, the informed trader selects a LO because of its price improvement. Furthermore, the uniformed trader's decision depends again on the level of information asymmetry. Consequently, in the case of high volatility, there are two possible optimal strategies: (BMO, SMO, BLO, SLO) and (BMO, SMO, NT, NT). The strategy (BMO, SMO, BLO, SLO) is

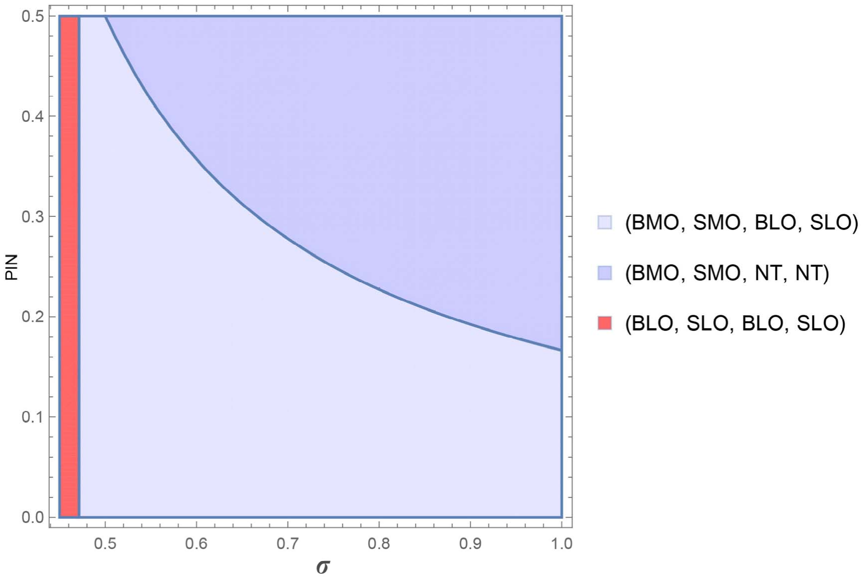

Figure 3: Optimal strategies at $t = 1$ in the single-venue market. Parameters values: $k_1 = 6$, $\lambda = 0.5$, $\tau = 0.05$, $\delta = 0.95$.

optimal when the degree of information asymmetry is sufficiently low (i.e., $PIN < \psi^U_{LO-NT}$), while (BMO, SMO, NT, NT) is optimal when the degree of information asymmetry is sufficiently high (i.e., $PIN \geq \psi^U_{LO-NT}$). We call these stocks *"High-Low"*, and *"High-High"*, respectively.

Figure 3 illustrates the optimal strategies in the first trading period of the single-venue market.[20] Numerical simulations show that: (i) the higher the asset's volatility is, the lower the probability of informed trading needs to be for an uninformed trader to choose NT. (ii) the strategy profile (BLO, SLO, NT, NT) is possible only for very specific parameter configurations, such as for $\pi > 0.5$. For this purpose, in the Internet Appendix IV, we show in Figure IV.2 the optimal strategies at $t = 1$ for parameter values that display the four possible equilibria.

The results derived in Proposition 1 are consistent with the previous work by Goettler et al. (2009), who show that informed traders switch from supplying to demanding liquidity when volatility changes from low to high.[21] Interestingly, our model encompasses both the model of Zhu (2014) and Buti et al. (2017). Note that when fundamental volatility and the PIN are high, that is, a *"High-High"* stock, the optimal strategy for an informed trader at $t = 1$ is to place a MO as in Zhu (2014). Similarly, when the probability of having an informed trader is very small ($\pi \to 0$), the model is similar to that in Buti et al. (2017), in which there is no asymmetric information. Notice also that when the asset's

[20] Figure IV.1 in the Internet Appendix IV shows a similar figure for a liquid market.

[21] Goettler et al. (2009) point out that first as volatility increases, the risk of a LO increases, as they are more likely to be picked-up for trading. Second, as volatility increases, so does the likelihood of finding mispriced orders in the LOB.

volatility is low and $\pi \to 0$, traders choose LO at $t = 1$, so the prevailing equilibrium is similar to $\mathcal{E}_3^{ND}$. Note that a *"High-Low"* stock corresponds to equilibrium $\mathcal{E}_1^{ND}$; a *"High-High"* to $\mathcal{E}_2^{ND}$; a *"Low-Low"* to $\mathcal{E}_3^{ND}$; and a *"Low-High"* to $\mathcal{E}_4^{ND}$.

The following corollary describes the comparative statics of κ^I_{MO-LO} and ψ^U_{LO-NT} with respect to various market and trader characteristics.

Corollary 1 *Ceteris paribus, κ^I_{MO-LO} increases with δ and k_1, and decreases with λ, while ψ^U_{LO-NT} increases with k_1, and decreases with λ and κ.*

The corollary above implies that for the informed trader at $t = 1$, an increase in the discount factor, a decrease in the liquidity of the asset $(1/k_1)$, or an increase in the probability that a liquidity trader arrives at $t = 2$ (ceteris paribus) reduces the relative attractiveness of a MO compared to a LO for an informed trader.[22] Regarding the uninformed trader at $t = 1$, a decrease in the liquidity of the asset, an increase in the probability that a liquidity trader arrives at $t = 2$, or a reduction in the volatility of the asset at $t = 2$ increases the attractiveness of a LO with respect to NT.

Note that according to Corollary 1, the condition $\sigma < \kappa^I_{MO-LO}\tau$ can be satisfied, ceteris paribus, for a low fundamental volatility or low liquidity stock (high k_1), or when rational traders are characterized by low immediacy (high δ) or participate as a relatively small proportion of the market (small λ). In addition, note that our classification of high/low fundamental volatility stocks also depends on the tick size: ceteris paribus, as the tick size increases, the low fundamental volatility region expands. To sum up, the characterization of stocks as *"High"* and "*Low"* in terms of liquidity, immediacy, or the proportion of rational traders gives analogous results to the characterization in terms of *"High"* and "*Low"* fundamental volatility. For simplicity, we illustrate our results by discussing them in terms of the fundamental asset volatility, but a similar analysis is possible by studying changes in other stock market and trader characteristics.

3 Equilibrium in the two-venue market model

We next consider a two-venue market model in which rational traders have access to both the exchange and the DP. Hence, the orders they can submit are given in (1).

[22] The informed trader's profits at $t = 1$ do not depend on the probability that an informed trader arrives in the next trading period, $\lambda\pi$. This is because an informed trader that submits a LO at $t = 1$ knows that the LO will not be executed in the next trading period against an order submitted by an informed trader, since an informed trader at $t = 2$ chooses an order of the same sign as the initial order. In addition, an informed trader at $t = 1$ correctly predicts that an uninformed trader at $t = 2$ never submits a MO of the opposite sign as the informed trader at $t = 1$.

The decision to submit an order to the DP depends on its probability of execution in this venue. We denote θ_t^I and θ_t^U as the probability of execution of a DO at trading period t for an informed and uninformed trader, respectively. In our model, the probability of execution in the DP is exogenous in the first trading period and it depends on the order imbalance in the DP. However, in the second trading period, this probability is endogenous since it depends on the order imbalance as well as on the traders' actions at $t = 1$.

As in the previous section, we solve the model backwards. Comparing the expected profits of each of the possible orders for each type of rational trader at $t = 2$ and $t = 1$, Lemma 2 states the strategies that are dominated, and hence, never chosen by a rational player.

Lemma 2 *In equilibrium, the following results hold:*

At $t = 2$, an informed trader may submit a MO or a DO, but never chooses a LO or NT, while an uninformed trader may submit either a MO, a DO, or NT, but never chooses a LO.

At $t = 1$, an informed trader may submit either a MO, a LO or a DO, but never chooses NT, while an uninformed trader may submit either a LO or NT, but never chooses a MO or a DO.

We find that in the second trading period, a LO is never chosen since it is never executed: a) if the LOB changed in the first period, then no MO arrives at the end of the second trading period, and hence, a LO has zero probability of execution; b) if the LOB did not change, then a LO can only be executed if an uninformed trader at $t = 1$ chooses a DO, but this cannot occur in equilibrium since the expected profits are null since an uninformed trader at $t = 1$ expects null profits of a DO in any event, executed or not.[23] Moreover, an informed trader at $t = 2$ never chooses NT since it is always dominated by a MO.

In the first trading period, an informed trader never chooses NT since it is always dominated by at least a MO. Moreover, the expected profits of a DO submitted by an informed trader at $t = 1$ are strictly positive (see Table C.3 in the Appendix C), and hence, a DO might be optimal for the informed trader at $t = 1$. By contrast, an uninformed trader at $t = 1$ may choose between a LO or NT since the expected profits of a MO are negative and those of a DO are null (see Table C.4 in the Appendix C).[24]

[23]The uninformed trader at $t = 2$ forms the correct beliefs that if a LO is executed at the end of this trading period, then his counterparty must be the informed trader who arrived at $t = 1$ with probability 1. However, this information reveals to the uninformed buyer (seller) that the value of the asset must be low (high), and hence, the expected profits of a LO are negative.

[24]Note that the mechanism is similar to those in Menkveld et al. (2017) and Brolley (2020): investors weigh each

Hence, the sustainable candidate strategy profiles at $t = 1$ as a PBE are:

$$\mathcal{E}_1^D : (BMO,\ SMO,\ BLO,\ SLO), \quad \mathcal{E}_2^D : (BMO,\ SMO,\ NT,\ NT),$$
$$\mathcal{E}_3^D : (BLO,\ SLO,\ BLO,\ SLO), \quad \mathcal{E}_4^D : (BLO,\ SLO,\ NT,\ NT),$$
$$\mathcal{E}_5^D : (BDO,\ SDO,\ BLO,\ SLO), \quad \mathcal{E}_6^D : (BDO,\ SDO,\ NT,\ NT).$$

Next, we describe the equilibrium of the trading game in the two-venue market.

Proposition 2 *In the two-venue market:*

Case A. *Suppose $k_1 > 1$. Then, we have the following cases:*

Case A.1 *If $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BLO, SLO, BLO, SLO) & \text{if } \theta_1^I \text{ is sufficiently small,} \\ (BDO, SDO, BLO, SLO) & \text{if } \theta_1^I \text{ is sufficiently large.} \end{cases}$$

Case A.2 *If $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN \geq \psi^U_{LO-NT}$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BLO, SLO, NT, NT) & \text{if } \theta_1^I \text{ is sufficiently small,} \\ (BDO, SDO, NT, NT) & \text{if } \theta_1^I \text{ is intermediate,} \\ (BDO, SDO, BLO, SLO) & \text{if } \theta_1^I \text{ is sufficiently large.} \end{cases}$$

Case A.3 *If $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN < \psi^U_{LO-NT}$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BMO, SMO, BLO, SLO) & \text{if } \theta_1^I \text{ is sufficiently small,} \\ (BDO, SDO, BLO, SLO) & \text{if } \theta_1^I \text{ is sufficiently large.} \end{cases}$$

Case A.4 *If $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^U_{LO-NT}$, then the optimal strategies profile at $t = 1$ are:*

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if } \theta_1^I \text{ is sufficiently small,} \\ (BDO, SDO, NT, NT) & \text{if } \theta_1^I \text{ is intermediate,} \\ (BDO, SDO, BLO, SLO) & \text{if } \theta_1^I \text{ is sufficiently large.} \end{cases}$$

Case B. *Otherwise, if $k_1 = 1$ (the asset is very liquid), then the optimal strategy profiles at $t = 1$*

order's execution risk against the price impact. However, in our model, price impact or execution risk are endogenously determined by optimal trading strategies at $t = 1$ and $t = 2$ as traders learn from the LOB.

are:

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta_1^I \text{ is sufficiently small,} \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta_1^I \text{ is sufficiently large.} \end{cases}$$

The proof in Appendix C characterizes the threshold values of θ_1^I for which each strategy profile is optimal at $t = 1$.

For Cases A and B, at t=2 an informed trader chooses either a MO or a DO, depending on the execution risk in the DP, while an uninformed trader chooses either a MO, a DO, or NT, depending on the execution risk in the DP, his beliefs and market conditions.

We start backwards by discussing the second trading period. An informed trader submits a *MO* (a *DO*) for all states of the *LOB* when the execution risk in the *DP* is sufficiently high (low) in relation to the price improvement in the new venue. As the execution risk in the *DP* decreases, an informed buyer replaces a *BMO* with a *BDO* in the following order according to the state of the *LOB*: $(A_1^2, B_1^1), (A_1^1, B_1^2), (A_1^1, B_1^1), (A_1^1, B_1^1 + \tau), (A_1^1 - \tau, B_1^1)$. This occurs because when a *BMO* was submitted at $t = 1$ and, hence, the best prices in the book are $\left(A_1^1, B_1^1\right)$, the gain from another *BMO* is the smallest in relation to a *BDO* even though the execution risk in the *DP* is relatively high. However, when a *SLO* was submitted at $t = 1$ and, hence, the best prices in the book are $(A_1^1 - \tau, B_1^1)$, the gain from a *BMO* is the largest in relation to that from a *BDO* despite the fact that the execution risk in the *DP* is relatively low.

For an uninformed trader at $t = 2$, the optimal strategy depends critically on his beliefs about the probability that a *MO*, a *LO* or a *DO* order was submitted by an informed trader at $t = 1$. When the state of the *LOB* contains no information; that is, (A_1^1, B_1^1), then an uninformed trader at $t = 2$ chooses *NT* since the expected profits of a *MO* are negative, and the expected profits of a *DO* are zero because the midpoint price is equal to the unconditional expected liquidation value of the asset. However, an uninformed trader may also choose a *MO* or a *DO* in the second trading period if the *LOB* conveys good news to the trader about the fundamental value of the asset. Note that, in contrast to the first trading period, if the probability of execution in the *DP* at $t = 2$ is sufficiently high, then an uninformed trader may migrate to the *DP*.

In the first trading period, Proposition 2 shows that having access to a *DP* may change the optimal submission strategy profiles for informed and uninformed traders. When the probability of execution in the *DP* for informed traders at $t = 1$ (i.e., θ_1^I) is sufficiently high, an informed trader switches trading venue, from the exchange to the *DP*. Otherwise, if the corresponding probability of execution in the DP is sufficiently low, then an informed trader submits the same types of orders to the exchange (MO or LO) as in the single-venue market. The threshold values of the probability of execution in the DP reflect the price improvement and execution trade-off of each order type. In case of execution, the best price is achieved by a LO, followed by a DO, and the worst price is given by a MO. While a LO has execution risk, the MO and DO do not face execution risk for an informed trader. Note that at $t = 1$, a DO faces no risk of execution since we find that if the DO is not executed in the first trading period, then the informed trader routes it back to the exchange as a MO at the end of the second trading period. However, when this order returns to the exchange, it faces the risk that the price worsened because of the order submitted by the trader that arrives at $t = 2$.

Although an uninformed trader never goes to the DP in the first trading period, as presented in Lemma 2, Proposition 2 shows that the existence of the DP might change the optimal strategy of an uninformed trader when the probability of execution in the DP for an informed trader is high enough, since an uninformed trader may switch from NT to a LO. This is because the low execution risk in the DP encourages an informed trader at $t = 2$ to trade in the DP rather than in the exchange. Consequently, the adverse selection that the uninformed trader faces at $t = 1$ in the exchange is lower than in the single-venue market, where the informed trader always chooses a MO. This makes the uninformed trader to trade.

Proposition 2 suggests that restricting the informed trader to participate in the DP might harm the uninformed trader. To illustrate this point, notice that Cases A.2 and A.4 of this proposition show that a significant reduction of θ_1^I might discourage the uninformed trader from participating in the exchange in the first trading period.

Figure 4 illustrates the optimal strategies at $t = 1$ with respect to the fundamental asset's volatility and information asymmetry for several values of θ_1^I, shown in Panels a), b), c), and d), respectively.[25] In Panel a), the graph has the same features as in Figure 3 since for small values of θ_1^I there is no migration to the DP. In Panel b) and c) we notice that there is a region of parameters in which orders migrate to the DP. Thus, (BLO, SLO, BLO, SLO), (BMO, SMO, BLO, SLO) or (BMO, SMO, NT, NT) prevail when PIN is high and the execution probability in the dark is relatively low. As this probability increases, there is migration to the dark - either (BDO, SDO, BLO, SLO) or (BDO, SDO, NT, NT) prevail, depending on the initial conditions. Notice that as the fundamental volatility increases, the informational advantage of the informed trader becomes higher, and hence, this trader has more

[25] Figure IV.3 in the Internet Appendix IV shows similar figures for a liquid market.

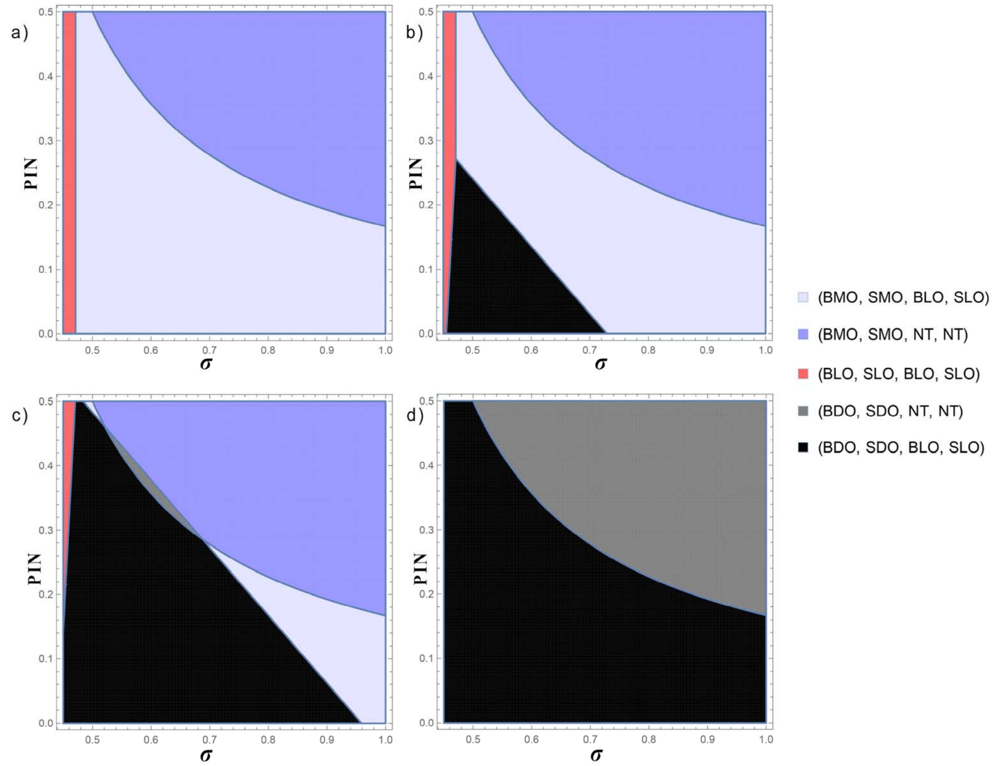

Figure 4: Optimal strategies at $t = 1$ with DP. Parameters values: $k_1 = 6$, $k_2 = 7$, $\lambda = 0.5$, $\tau = 0.05$, and $\delta = 0.95$. In Panel a) $\theta_1^I = 0.05$, in Panel b) $\theta_1^I = 0.10$, in Panel c) $\theta_1^I = 0.13$, and in Panel d) $\theta_1^I = 0.25$.

incentives to trade immediately. This is because the price improvement of a DO does not compensate the risk of not being executed in the DP at $t = 1$ and returning to the LOB, where the price might have worsened. In Panel d), we find that the informed trader fully migrates to the DP at $t = 1$, while the uninformed trader decides not to trade whenever the adverse selection he faces is high enough (i.e., when PIN is high enough and the fundamental volatility is not low).[26]

4 Market Performance

In this section we examine how the existence of a DP affects market performance. To do so, we compare several measures of market quality of the two-venue market in relation to the single-venue market: price informativeness, expected inside spread, and rational traders' expected profits. We use the stock categorization with respect to fundamental asset volatility and information asymmetry

[26] In addition to the parameter values defined in the caption of Figure 4, we assume that the beliefs at $t = 2$ are such that an uninformed buyer (seller) does not select a BLO (SLO) when there is no change in the LOB prices, and that an informed buyer (seller) chooses a BMO (SMO) at $t = 2$ when the LOB has not changed. Furthermore, we assume that the probabilities of execution of a DO when the order imbalance is of size 2 submitted by either an informed or uninformed trader at $t = 1$ is equal to zero.

defined in Section 2, but we can obtain the same empirical implications with respect to initial stock liquidity, traders' immediacy, or rational traders' participation rate. Furthermore, we highlight some of the implications of our model that may be tested in applied work. These implications are relevant for the current policy and regulatory debate on the effects of dark trading on price informativeness, order flow fragmentation and market liquidity. The first three subsections focus on market performance in the first trading period, while subsection 4.4 discusses the effects in the second trading period.

4.1 Price Informativeness

Price informativeness is at the heart of the regulatory debate about whether DPs increase or decrease price discovery. We measure price informativeness in a given trading period t as the reduction in variance of the liquidation value of the asset after observing the set of best ask and bid prices right after finishing the trading process in which a new trader is involved in this trading period.

Proposition 3 (Price informativeness) *In the two-venue market, price informativeness is lower in the first trading period if there has been order migration to the DP.*

The implications regarding price informativeness are a consequence of the order flow segmentation results. In the first trading period, dark trading harms price informativeness since the DP is only attractive to informed traders. This is consistent with the existing empirical results of Hendershott and Jones (2005); Comerton-Forde and Putniņš (2015), when the proportion of dark trading is above 10%; Hatheway et al. (2017), and Brogaard and Pan (2019). With regards to the order flow segmentation, Naes and Odegaard (2006) find that there is informational content in crossing network trades, while Nimalendran and Ray (2014) find that informed traders strategically use both crossing networks and exchanges.

4.2 Market liquidity

The next proposition shows how access to a DP affects market liquidity, measured by the ex-ante expected inside spread in the exchange.

Proposition 4 (Expected inside spread) *When there is order migration to the DP in the first trading period, the expected inside spread is lower (higher) in the two-venue market for high (low) fundamental volatility stocks.*

Proposition 4 indicates that in the first trading period, for high fundamental volatility stocks, if the existence of a DP makes the informed trader switch from a MO to a DO, then the expected inside spread decreases, regardless of the behavior of the uninformed trader. We can explain these results by simply noting that the switch from a MO to a DO reduces the inside spread because the trader does not consume liquidity in the exchange.[27] Therefore, for high volatility stocks our results are consistent with empirical studies that show that DP trading increases market liquidity (Gresse, 2006; Buti et al., 2011; Ready, 2014).[28]

In contrast, for low fundamental volatility stocks, the switch from a LO to a DO increases the inside spread. This is because the trader does not supply liquidity to the exchange, whereas the switch from NT to a LO reduces the inside spread, as in this case, the trader supplies liquidity to the exchange. Hence, for the *"Low-High"* stocks, a potential ambiguity arises in the transition from $\mathcal{E}_4^{ND}$ to $\mathcal{E}_5^{D}$. However, note that $\mathcal{E}_4^{ND}$ prevails only if the probability that an informed trader arrives is sufficiently high $\left(\pi > \frac{1}{2}\right)$. Hence, the effect of the informed trader on the inside spread dominates the effect of the uninformed trader, and therefore, the expected inside spread in the two-venue market is unequivocally larger than in the single-venue market. These results for low volatility stocks are consistent with those empirical studies that show that the existence of a DP decreases market liquidity (Nimalendran and Ray, 2014; Weaver, 2014; Kwan et al., 2015; Degryse et al., 2015; Hatheway et al., 2017). Finally, when there is no migration to the DP, the spread stays the same. Foley and Putniņš (2016) show that midpoint dark trading in the Canadian market does not benefit or harm market liquidity, and Gresse (2017) shows that dark trading is not harmful to any dimension of market liquidity.

Our results are also related to the empirical work that studies how the effects of a DP vary with the tick size. Ceteris paribus, we see that as the tick size increases, the low volatility region expands (where in the first trading period the informed trader submits a limit order when the DP is not available). This result implies that markets with a high tick size are more likely to see an increase in the expected inside spread when there is order migration to the DP. This result is similar to that reported by Buti et al. (2015), who show that allowing DOs to "queue-jump" displayed orders reduces traders' willingness to display LOs on competing lit markets. Our results are also consistent with Buti et al. (2011) and Kwan et al. (2015), who show that when spreads on traditional exchanges are constrained

[27]Note that in cases $A.3$ and $A.4$ the uninformed trader either does not change his order type or changes from NT to LO. This last change also reduces the inside spread.

[28]Our results for high fundamental volatility stocks are in line with the theoretical conjecture by Buti et al. (2017) that dark trading would not necessarily cause a wider spread even under asymmetric information. However, our results differ for low fundamental volatility stocks.

by minimum pricing increments, traders have incentives to migrate to a DP since the execution risk in the DP is lower than that for LOs in the exchange.[29]

4.3 Expected profits

In what follows, we compare the unconditional expected profits of rational traders in the two-venue market in relation to the single-venue market.

Proposition 5 (Expected profits) *In the two-venue market, the migration of an order to the DP in the first trading period leads to strictly larger profits for informed traders, and higher or equal profits for uninformed traders relative to the single-venue market.*

The expected profits of each type of rational trader are not lower in the two-venue market compared to the single-venue market. First, an informed trader strictly increases his profits when choosing a DO since the price improvement obtained by submitting a DO outweighs the execution risk in the DP. Second, even if an uninformed trader does not go to the DP initially, he has larger profits in the two-venue market under certain market conditions. This is because the migration of the informed trader's orders to the DP reduces adverse selection in the LOB. Consequently, the probability that an uninformed trader faces an informed trader is smaller in the two-venue market. Therefore, the uninformed trader's expected profits are higher or equal in the two-venue market compared to the single-venue market. Moreover, when the uninformed trader switches from NT to a LO, his profits are strictly larger in the two-venue market.

4.4 Market performance at t=2

In the previous subsections, we have focused on the effects of the coexistence of a dark pool alongside an exchange in the first trading period. Interestingly, traders decisions in the second period depend on the previous decisions (reflected both in the limit order book and in the endogenous execution probability in the dark). However, since the game ends right after the second period, the execution probability of a LO at $t = 2$ is zero. As a result, we cannot disentangle the end of game effects from the (endogenous) effects brought about by $t = 1$ optimal decisions. Hence, we discuss briefly each market indicator and to further clarify some results, we include some examples.

[29] Note that our results are similar, but the mechanism is different. In our model, the tick size does not affect the execution probability in the DP, but it affects the profits obtained in case of execution.

The effects of the existence of the DP on *price informativeness* in the second period depend crucially on market conditions. When market conditions are such that there is migration to the DP initially, the price informativeness is always lower in the two-venue market. In contrast, when there is no migration to the DP initially, we might have that price informativenes can be both higher or lower in the two-venue market than in the single-venue market depnding on how order flows gets segmented. The initial market conditions determine whether there is or not order flow segmentation in the second period (since in this period both the informed and uninformed trader might submit orders to the DP if conditions are favorable). The change in the DP's order attractiveness for uninformed traders between the first and the second trading period brings about differences in how the coexistence of the DP with the LOB affects price informativeness.

Let us consider the case of a *"High-High"* stock to exemplify these differences. When the initial execution probability in the dark pool is low, in equilibrium at $t = 1$ an informed trader chooses a MO and an uninformed trader NT both in the single-venue and two-venue market; that is, $\mathcal{E}_2^{ND}$ and $\mathcal{E}_2^{D}$. When there is no change in traders' behavior at $t = 2$, then price informativeness stays the same. However, if at $t = 2$ informed and uninformed traders choose different trading venues, then we have contrasting results regarding price informativeness in the second trading period. Thus, if there is segmentation of the order flow such that the informed trader goes to the DP and the uninformed trader remains in the exchange, then we expect a lower price informativeness in the two-venue market. By contrast, when there is segmentation of the order flow but the informed trader stays in the exchange and the uninformed trader migrates to the DP, then we expect a higher price informativeness if there is access to a DP.[30] The key variables that determine the traders' behavior and lead to an increase or a decrease in price informativeness in this example are the execution probabilities in the DP at $t = 2$ and the price improvement obtained in the DP. However, stock market and trader characteristics are also important in the magnitude of this change (for instance, price informativeness increases with fundamental volatility, as does the increase/decrease in price informativeness due to the coexistence of the DP with the exchange).

With respect to *market liquidity* in the second trading period, we note that at the beginning of the second trading period, we could have different spreads depending on whether the DP is available or not. Thus for high volatility stocks, we expect that the inside spread at the beginning of $t = 2$

[30] When the order flow segmentation is such that informed traders concentrate in the exchange while uninformed traders use the DP, our results are consistent with the empirical work of Ready (2014) and Comerton-Forde and Putniņš (2015) (for low levels of dark trading).

in the two-venue market to be lower or to stay the same (see Proposition 3). In these cases, having access to the DP unambiguously reduces the ex-ante expected inside spread. This is because at $t = 2$, an informed trader might switch from a MO to a DO, and an uninformed trader from a MO or NT to a DO, which reduces the expected inside spread. However, we can obtain different results for low volatility stocks for which we expect, in equilibrium, a higher inside spread at the beginning of the second trading period in the two-venue market. Thus, the possibility of submitting a DO instead of a MO in the second trading period might reduce the inside spread in the two-venue market, which goes in the opposite direction to the one obtained in the first trading period.

With respect to *expected profits* in the second trading period, the informed trader's expected profits are not lower in the two-venue market. However, the changes in the uninformed trader's expected profits depend on the market conditions. A priori one would expect that when uninformed traders have the opportunity to trade in the DP their welfare increases. But from Proposition 3, we know that uninformed traders can extract less information from the book in the two-venue market if there is order migration, which makes them worse off. For *"High-Low"* stocks the uninformed trader's expected profits are always larger in the two-venue market. This is because in this market structure uninformed traders go to the DP in some states of the book, while in the single-venue market, uninformed traders do not trade. For the other types of stocks, the uninformed trader's expected profit are higher except when the PIN is high and the execution probability in DP of uninformed, θ_2^U, is small. In this situation, it is not beneficial for an uninformed trader to leave the exchange and migrate to the DP. However, if the execution probability in DP, θ_2^U, is high then the expected profits of the uninformed trader are higher in the two-venue market.

5 Policy implications

Our work can also inform the regulatory debate on DPs. We use our analysis to formulate an additional policy related prediction. The European Commission aims to limit dark trading through the Double Volume Cap (DVC) mechanism as part of MiFID II/MiFIR, that was implemented in 2018. The DVC introduces a cap on dark trading that limits the trading volume of a financial instrument in any single DP to 4% of its total volume of trading in the previous year. We can think of this cap on dark trading as equivalent to an upper limit on the execution probability in the DP. It is worth stressing that if the cap on dark trading is binding such that the informed trader is restricted to trading in the exchange instead of trading in the DP in the first trading period, then the DVC limits both the benefits and the drawbacks of the coexistence of a DP and an exchange. In particular, our analysis implies that imposing a binding cap on dark trading has the following effects on market quality in the first trading period: (i) price informativeness is higher; (ii) the expected inside spread is higher for high fundamental volatility stocks and lower for low fundamental volatility stocks.

Our results show that while the DVC is expected to have a positive effect on price informativeness in the first trading period, its effects on the other market quality parameters depend on stock and trader characteristics. Specifically, the DVC policy might have unintended negative consequences, such as decreasing liquidity for high volatility stocks or decreasing the profits for rational traders. To the best of our knowledge, the only study that examines the consequences of DVC is Johann et al. (2019). They support our view that the consequences of the ban may not be the ones that regulators expected. However, they note that with the implementation of the ban, trading volume migrated from DPs to "quasi-dark" trading mechanisms rather than back to exchange markets (three times more volume went to these "quasi dark venues", which are alternative venues that do not exist in our model). They also find a negligible impact of DP caps on market liquidity and short-term price efficiency. ESMA (2019, 2020) concluded that the MiFID II measures in place since January 2018 had failed to curb trading in DPs and consequently they recently decided to transform the double volume cap mechanism by removing the volume cap that limits dark trading in a single venue at 4%.

6 Concluding Remarks

This paper examines the impact of an opaque dark pool that competes with a transparent exchange organized as a limit order book in a model with asymmetric information about the liquidation value of the asset. We find that the effects of this competition on price informativeness and market performance depend critically on the stock categorization in terms of high/low fundamental volatility and high/low information asymmetry as these factors are determinant when selecting the venue and the type of order. As a result, regulators should take into account the market conditions in the implementation of policies that aim to curb dark trading.

The existing empirical research often gives conflicting results on the effects of the presence of a DP alongside an exchange. Studies differ in their research questions, the type of data, and regulatory environments. Thus, most of these empirical studies suggest that the discrepancies are driven by differences in the market structure and financial regulations. Interestingly, our analysis predicts that the coexistence of the two venues may have both negative and positive effects on the market performance of the LOB, even if the market structure and regulatory environment are exactly the same. As our previous analysis shows, stock and trader characteristics affect the optimal order submission strategies, and in turn, these have implications for market quality and traders' profits. Moreover, our policy analysis concludes that regulators should consider that establishing measures to reduce informed traders' participation in DPs could have unintended negative consequences on other traders and on some market performance indicators.

Future work could extend our theoretical model in different ways, such as by considering the case in which the initial prices of the limit order book are asymmetrically located in the grid. Allowing for this asymmetry might induce an uninformed trade to migrate to the DP initially. Another interesting theoretical generalization of our model may include modelling the execution probability in the dark as an increasing function of fundamental volatility or fully endogenizing the behavior of the dark pool liquidity provider. Modelling the execution probability in the dark pool as an increasing function of fundamental volatility (as the empirical literature suggests) seems not to affect the existence of the equilibria. However, it varies the thresholds at which all traders execute their orders in the exchange and those equilibria where there is migration to the dark pool. When the execution probability does not depend on volatility, a market order is more desirable for informed traders as a high volatility increases their informational advantage. However, if the execution probability in the dark pool increases also with fundamental volatility, the dark order becomes more profitable for high volatility stocks (relative to the constant case). Additionally, our results call for the development of applied work studying the effects of asymmetric information in the competition between trading venues with different degrees of transparency on market quality and traders' profits.

Appendices

Appendix A is a summary of the notation used; Appendix B shows the proofs related to the single venue market model (Lemma 1 and Proposition 1); and Appendix C includes the proofs related to the two venue market model (Lemma 2, Lemma C.1 and Proposition 2). The rest of the proofs and further details of the calculations underlying our results can be found in the Internet Appendices.

A Notation summary

This appendix summarizes the key notations used in the paper.

LOB denotes the limit order book, DP denotes the dark pool, ND denotes the single-venue market, and D denotes the two-venue market.

Types of Traders

Type	Definition
R	Rational trader, $R \in \{I, U\}$
I (IH/IL)	Informed trader (who observes a high/low liquidation value)
U (UB/US)	Uninformed trader (who buys/sells)
LT	Liquidity trader

Types of Orders

Type	Definition
MO (BMO/SMO)	Market order (Buy/Sell market order)
LO (BLO/SLO)	Limit order (Buy/Sell limit order)
DO (BDO/SDO)	Dark order (Buy/Sell dark order)
NT	No trade

Exogenous Parameters

Parameters	Definition
$\widetilde{V}$	Liquidation value of the asset, may take two values $V \in \left\{V^H, V^L\right\}$
μ and σ	The unconditional mean and volatility of the liquidation value $\widetilde{V}$
A_1^p, B_1^p	Ask and bid prices at time $t = 1$ and position p
τ	Tick size
k_p	A natural number such that $A_1^p = \mu + k_p\tau$ $(B_1^p = \mu - k_p\tau)$
κ	A real number such that $\sigma = \kappa\tau$
λ	Probability that a rational trader arrives to the market
π	Probability that a rational trader is informed
PIN	Probability that an informed trader arrives to the market which is equal to $\pi\lambda$
δ	Discount factor (immediacy) of rational traders
θ_1^R	Probability of execution of a DO at $t = 1$ for a rational trader

Endogenous Variables

Variable	Definition
A_2^p, B_2^p	Ask and bid prices at time $t = 2$ and position p
θ_2^R	Probability of execution of a DO at $t = 2$ for a rational trader of type R
$\Pi_{O,t}^R$	Profit from trading for a trader of type R using order O at date t
κ_{MO-LO}^I	Volatility threshold for informed trader's decision between MO and LO
ψ_{LO-NT}^U	PIN threshold for uninformed trader's decision between LO and NT

B Single-venue market model

Definition B.1 *Let us define Ω_o and Γ_o as the probability that an informed trader and uninformed trader at $t = 1$ choose an order $\mathcal{O} \in \mathbb{O}_{ND}$, where $o = 0$ corresponds to a NT order; $o = 1$ to a MO; $o = 2$ to a LO; and such that $\sum_{o=0}^{2} \Omega_o = 1$ and $\sum_{o=0}^{2} \Gamma_o = 1$.*

Proof of Lemma 1. We solve the game backwards. At $t = 2$, the expected profits for an informed and uninformed buyer and seller are: Note that at $t = 2$ the expected profits of each strategy depend

State of the book	**IH**			**IL**		
	BMO	*BLO*	*NT*	*SMO*	*SLO*	*NT*
(A_1^1, B_1^1)	$(\kappa - k_1)\,\tau$	0	0	$(\kappa - k_1)\,\tau$	0	0
(A_1^2, B_1^1)	$(\kappa - k_2)\,\tau$	0	0	$(\kappa - k_1)\,\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(\kappa - k_1)\,\tau$	0	0	$(\kappa - k_1 + 1)\,\tau$	0	0
(A_1^1, B_1^2)	$(\kappa - k_1)\,\tau$	0	0	$(\kappa - k_2)\,\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$(\kappa - k_1 + 1)\,\tau$	0	0	$(\kappa - k_1)\,\tau$	0	0

Table B.1: Expected profits of an informed buyer (IH) and an informed seller (IL) at $t = 2$ when traders do not have access to the dark pool.

State of the book	**UB**			**UB**		
	BMO	*BLO*	*NT*	*SMO*	*SLO*	*NT*
(A_1^1, B_1^1)	$-k_1\tau$	0	0	$-k_1\tau$	0	0
(A_1^2, B_1^1)	$(X\kappa - k_2)\,\tau$	0	0	$-(k_1 + X\kappa)\,\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(Y\kappa - k_1)\,\tau$	0	0	$-(k_1 - 1 + Y\kappa)\,\tau$	0	0
(A_1^1, B_1^2)	$-(X\kappa + k_1)\,\tau$	0	0	$(X\kappa - k_2)\,\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$-(Y\kappa + k_1 - 1)\,\tau$	0	0	$(Y\kappa - k_1)\,\tau$	0	0

Table B.2: Expected profits of an uninformed buyer (UB) and an uninformed seller (US) at $t = 2$ when traders do not have access to the dark pool.

on the state of the LOB (which depends on the chosen strategy at $t = 1$). Uniformed traders at $t = 2$ form beliefs about the strategies and type of player at $t = 1$. Thus, we define the uninformed traders' belief at $t = 2$ about the probability that the MO (observed in the LOB) was submitted by an informed trader as

$$X = \frac{\lambda\pi\Omega_1}{1 - \lambda + \lambda\pi\Omega_1 + \lambda(1 - \pi)\Gamma_1}. \tag{B.1}$$

Similarly, we define the uninformed traders' belief at $t = 2$ about the probability that the LO (observed in the LOB) was submitted by an informed trader as

$$Y = \frac{\pi\Omega_2}{\pi\Omega_2 + (1 - \pi)\Gamma_2}. \tag{B.2}$$

By comparing the expected profits of an informed trader at $t = 2$ we obtain that the informed trader always submits a MO. Similarly, we compare the profits of an uniformed trader and see that he never chooses to submit a LO. His choice between a MO or NT depends on the uninformed trader beliefs that the order placed at $t = 1$ that he observes in the book comes from an informed trader, as it can be seen in Table B.3.

State of the book	**UB**	**US**
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } X\kappa > k_2 \\ NT & \text{if } X\kappa \leq k_2 \end{cases}$	NT
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} MO & \text{if } Y\kappa > k_1 \\ NT & \text{if } Y\kappa \leq k_1 \end{cases}$	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } X\kappa > k_2 \\ NT & \text{if } X\kappa \leq k_2 \end{cases}$
$(A_1^1 - \tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } Y\kappa > k_1 \\ NT & \text{if } Y\kappa \leq k_1 \end{cases}$

Table B.3: Optimal trading strategies of an uniformed buyer (UB) and seller (US) at $t = 2$ when traders do not have access to the dark pool.

At $t = 1$, the expected profits of an informed and uniformed trader are presented in Table B.4 and

Table B.5, respectively.

IH	IL	Expected Profits
BMO	SMO	$(\kappa - k_1)\,\tau$
BLO	SLO	$\delta\frac{1-\lambda}{2}\,(\kappa + k_1 - 1)\,\tau$
NT	NT	0

Table B.4: Expected profits of an informed buyer (IH) and seller (IL) at $t = 1$ when traders do not have access to the dark pool.

UB	US	Expected Profits
BMO	SMO	$-k_1\tau$
BLO	SLO	$\frac{\delta}{2}\,((1-\lambda+\lambda\pi)\,(k_1-1) - \lambda\pi\kappa)\,\tau$
NT	NT	0

Table B.5: Expected profits of an uninformed buyer (UB) and seller (US) at $t = 1$ when traders do not have access to the dark pool.

It can be easily seen that at $t = 1$ the informed trader never chooses NT, while the uninformed never chooses a MO. ∎

Proof of Proposition 1. We follow the steps outlined in Internet Appendix II to check if a particular strategy profile constitutes a PBE.

Because of the symmetry of the model, without any loss of generality, at $t = 1$ we focus on buyers. We distinguish two cases: Case A ($k_1 > 1$) and Case B ($k_1 = 1$).

Case A. We present the full proof for one of the possible strategy profile at $t = 1$ that yields an equilibrium. The proofs of all the other 3 equilibria are sketched here and can be obtained on request from the authors.

$\mathcal{E}_1^{ND}$: (BMO, SMO, BLO, SLO)

First step. In this case $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = 0$, $\Gamma_0 = 0$, $\Gamma_1 = 0$, and $\Gamma_2 = 1$.

Second step. Using Bayes' rule we obtain that $X^{1,ND} = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$ and $Y^{1,ND} = 0$.

Third step. Applying Lemma 1, we know that at $t = 2$ the optimal strategy of informed traders is to choose a MO, while the optimal strategy of the uninformed trader is as follows:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \le k_2 \end{cases}$	NT
$(A_1^1, B_1^1 + \tau)$	NT	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \le k_2 \end{cases}$
$(A_1^1 - \tau, B_1^1)$	NT	NT

Table B.6: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

Fourth step. Given the optimal response of traders at $t=2$, we find the optimal action for all rational traders at $t=1$.

Informed traders at $t=1$ have no incentives to deviate from the prescribed strategy profile whenever

$$\kappa - k_1 \geq \delta \frac{1-\lambda}{2}\left(\kappa + k_1 - 1\right). \tag{B.3}$$

Uninformed traders at $t=1$ have no incentives to deviate from the prescribed strategy if and only if

$$\left(1-\lambda\right)\left(k_1-1\right) - \lambda\pi\left(\kappa - \left(k_1-1\right)\right) > 0. \tag{B.4}$$

Fifth step. Nobody at $t=1$ has unilateral incentives to deviate from (BMO, SMO, BLO, SLO) when both conditions (B.3) and (B.4) are satisfied, and these conditions can be rewritten as

$$\kappa^{I}_{MO-LO}\tau \leq \sigma \text{ and } PIN < \psi^{U}_{LO-NT},$$

where the expression of ψ^{U}_{LO-NT} and κ^{I}_{MO-LO} are given in the statement of this proposition.

Finally, combining Table B.6 and Expression (B.4), it follows that an uninformed trader always selects NT at $t=2$.

$\mathcal{E}^{ND}_{2}$: (BMO, SMO, NT, NT)

Note that $X^{2,ND} = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$ and $Y^{2,ND}$ is undetermined $Y^{2,ND} \in [0,1]$, we obtain that no trader at $t=1$ has unilateral incentives to deviate in $\mathcal{E}^{ND}_{2}$ whenever:

$$\begin{aligned} \kappa - k_1 &\geq \delta \frac{1-\lambda}{2}\left(\kappa + k_1 - 1\right) \text{ and} && \text{(B.5)}\\ 0 &\geq \left(1-\lambda\right)\left(k_1-1\right) - \lambda\pi\left(\kappa - \left(k_1-1\right)\right), && \text{(B.6)}\end{aligned}$$

which can be rewritten as $\kappa^{I}_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^{U}_{LO-NT}$.

Finally, in the following table we include the moves that are in the equilibrium path at $t=2$ for an uninformed trader, taking into account that (BMO, SMO, NT, NT) is the strategy profile chosen at $t=1$.

State of the book	**UB**	**US**
(A^1_1, B^1_1)	NT	NT
(A^2_1, B^1_1)	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$	NT
(A^1_1, B^2_1)	NT	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$

Table B.7: Optimal choice of uninformed traders at $t=2$ when the strategy profile at $t=1$ is (BMO, SMO, NT, NT)

$\mathcal{E}^{ND}_{3}$: (BLO, SLO, BLO, SLO)

Following the same procedure as above and noting that $X^{3,ND}=0$ and $Y^{3,ND}=\pi$, we obtain that no trader at $t=1$ has unilateral incentives to deviate in $\mathcal{E}^{ND}_{3}$ whenever:

$$\begin{aligned} \delta \frac{1-\lambda}{2}\left(\kappa + k_1 - 1\right) &> \kappa - k_1 \text{ and} && \text{(B.7)}\\ \left(1-\lambda\right)\left(k_1-1\right) - \lambda\pi\left(\kappa - \left(k_1-1\right)\right) &> 0, && \text{(B.8)}\end{aligned}$$

which can be rewritten as $\sigma < \kappa^{I}_{MO-LO}\tau$ and $PIN < \psi^{U}_{LO-NT}$.

Finally, in the following table we include the moves that are in the equilibrium path at $t = 2$ for an uninformed trader, taking into account the conditions that must be satisfied if (BLO, SLO, BLO, SLO) is the strategy profile chosen at $t = 1$.

State of the book	UB	US
(A_1^2, B_1^1)	NT	NT
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} MO & \text{if } \pi\kappa > k_1 \\ NT & \text{if } \pi\kappa \leq k_1 \end{cases}$	NT
(A_1^1, B_1^2)	NT	NT
$(A_1^1 - \tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } \pi\kappa > k_1 \\ NT & \text{if } \pi\kappa \leq k_1 \end{cases}$

Table B.8: Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, BLO, SLO)

$\mathcal{E}_4^{ND}$: (BLO, SLO, NT, NT)

Following the same procedure as above and noting that $X^{4,ND} = 0$ and $Y^{4,ND} = 1$, we obtain that no trader at $t = 1$ has unilateral incentives to deviate in $\mathcal{E}_4^{ND}$ whenever:

$$\delta \frac{1-\lambda}{2} (\kappa + k_1 - 1) > \kappa - k_1 \text{ and} \tag{B.9}$$

$$0 \geq (1-\lambda)(k_1 - 1) - \lambda\pi(\kappa - (k_1 - 1)), \tag{B.10}$$

which can be rewritten as $\sigma < \kappa_{MO-LO}^{I} \tau$ and $PIN \geq \psi_{LO-NT}^{U}$.

Finally, in the following table we include the moves that are in the equilibrium path at $t = 2$ for an uninformed trader, taking into account the conditions that must be satisfied if (BLO, SLO, NT, NT) is the strategy profile chosen at $t = 1$.

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	NT
$(A_1^1, B_1^1 + \tau)$	MO	NT
(A_1^1, B_1^2)	NT	NT
$(A_1^1 - \tau, B_1^1)$	NT	MO

Table B.9: Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, NT, NT).

Case B. We have to replace $k_1 = 1$ in the proof of Case A. It should only be noted that when $k_1 = 1$ the conditions (B.4) and (B.8) are never satisfied and, therefore, the strategy profiles at $t = 1$ (BMO, SMO, BLO, SLO) and (BLO, SLO, BLO, SLO) cannot be part of an equilibrium of the game. By contrast, when $k_1 = 1$, the conditions (B.6) and (B.10) are always satisfied. However, the condition (B.9) is never satisfied when $k_1 = 1$, and therefore, the strategy profile (BLO, SLO, NT, NT) cannot be either part of an equilibrium of the game.

C Two-venue market model

Definition C.1 *Let us define Ω_o and Γ_o as the probability that an informed trader and uninformed trader at $t = 1$ choose an order $\mathcal{O} \in \mathbb{O}_D$, where $o = 0$ corresponds to a NT order; $o = 1$ to a MO; $o = 2$ to a LO; $o = 3$ to a DO; and such that $\sum_{o=0}^{3} \Omega_o = 1$ and $\sum_{o=0}^{3} \Gamma_o = 1$.*

Definition C.2 *We define as $\mathbb{B}_1$ the set of all possible states of the LOB at the end of the first trading period and by $\mathcal{B}_1 \in \mathbb{B}_1$ a possible state of the book (see Internet Appendix I for a full definition). The state of the book $\mathcal{B}_1 = \varnothing$ is the state when the best prices at the end of the first trading period in the book are (A_1^1, B_1^1).*

Proof of Lemma 2. Note that the set of the possible states of the LOB is the same as in the case there is no DP. However, the state of the book (A_1^1, B_1^1) can be obtained either because a trader arrived and decided not to trade, or because a trader arrived and he submitted a DO.

We solve the model backwards. At $t = 2$ the expected profits of each strategy depend on the state of the LOB. Additionally, uninformed traders form beliefs about the strategies that have been chosen at $t = 1$. Let X and Y be defined as in (B.1) and (B.2), respectively, and Z denote the uninformed trader's belief at $t = 2$ about the probability that a DO that returns to the exchange as a MO at the end of the second trading period was submitted by an informed, which is equal to

$$Z = \frac{(1-\theta_1^I)\pi\Omega_3}{(1-\theta_1^I)\pi\Omega_3 + (1-\theta_1^U)(1-\pi)\Gamma_3}.$$

As in the case when the DP was not available, and without loss of generality, we will focus on the expected profits for an informed and an uformed buyer at $t = 2$, as summarized in Table C.1 and Table C.2, respectively.

Define P_I as the probability of execution of a limit order placed by an informed trader at $t = 2$ conditional on the fact that there is no change in the LOB during the first trading period, and equals

$$P_I = p_{BLO,2}^{IH}(\mathcal{B}_1 = \varnothing) = \frac{(1-\theta_1^U)\frac{1-\pi}{2}\Gamma_3}{\pi\Omega_3 + (1-\pi)(\Gamma_0+\Gamma_3)}.$$

IH	BMO	BDO	BLO	NT
(A_1^1, B_1^1)	$(\kappa - k_1)\tau$	$\theta_2^I \kappa\tau$	$P_I\delta(k_1+\kappa-1)\tau$	0
(A_1^2, B_1^1)	$(\kappa - k_2)\tau$	$\theta_2^I\left(\kappa - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1+\tau)$	$(\kappa - k_1)\tau$	$\theta_2^I\left(\kappa - \frac{1}{2}\right)\tau$	0	0
(A_1^1, B_1^2)	$(\kappa - k_1)\tau$	$\theta_2^I\left(\kappa + \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1-\tau, B_1^1)$	$(\kappa - k_1 + 1)\tau$	$\theta_2^I\left(\kappa + \frac{1}{2}\right)\tau$	0	0

Table C.1: Expected profits of an informed buyer (IH) at $t = 2$

Define P_U as the probability of execution of a limit order placed by an uninformed trader at $t = 2$ given that there are no changes in prices in the LOB during the first trading period, and equals

$$P_U = p_{BLO,2}^{UB}(\mathcal{B}_1 = \varnothing) = \frac{1}{2}\frac{(1-\theta_1^I)\pi\Omega_3 + \left(1-\theta_1^U\right)(1-\pi)\Gamma_3}{\pi\Omega_3 + (1-\pi)(\Gamma_0+\Gamma_3)}.$$

UB	BMO	BDO	BLO	NT
(A_1^1, B_1^1)	$-k_1\tau$	0	$P_U\delta(k_1 - Z\kappa - 1)\tau$	0
(A_1^2, B_1^1)	$(X\kappa - k_2)\tau$	$\theta_2^U\left(X\kappa - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1+\tau)$	$(Y\kappa - k_1)\tau$	$\theta_2^U\left(Y\kappa - \frac{1}{2}\right)\tau$	0	0
(A_1^1, B_1^2)	$-(X\kappa + k_1)\tau$	$-\theta_2^U\left(X\kappa - \frac{k_2-k_1}{2}\right)\tau$	0	0
$(A_1^1-\tau, B_1^1)$	$-(Y\kappa + k_1 - 1)\tau$	$-\theta_2^U\left(Y\kappa - \frac{1}{2}\right)\tau$	0	0

Table C.2: Expected profits of an uninformed buyer (UB) at $t = 2$

At $t = 1$ the expected profits of an informed IH and an uformed buyer UB are summarized in Table C.3 and Table C.4, respectively.[31]

[31]Notice that due to the symmetry of the game, the expected profits of the informed IL trader and uninformed seller US are the same as the ones displayed in Tables C.3 and C.4, respectively.

IH	Expected Profits
BMO	$(\kappa - k_1)\,\tau$
BLO	$\frac{\delta(1-\lambda)}{2}(\kappa + k_1 - 1)\,\tau$
BDO	$\theta_1^I \kappa\tau + (1-\theta_1^I)\delta\left(\lambda\frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} + (\kappa - k_1) - (k_2 - k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right)\tau$
NT	0

Table C.3: Expected profits of an informed buyer (IH) at $t = 1$

UB	Expected Profits
BMO	$-k_1\tau$
BLO	$\frac{\delta}{2}\left((1-\lambda)(k_1 - 1) - \lambda\pi I_{SMO,2}^{IL,\mathcal{B}_1=BLO}(\kappa - k_1 + 1)\right)\tau$
BDO	0
NT	0

Table C.4: Expected profits of an uninformed buyer (UB) at $t = 1$

where $I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}$ and $I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing}$ are indicator functions such that $I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} = 1$ if at $t = 2$, an US selects a SLO when the LOB has not changed at $t = 1$, and $I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} = 0$, otherwise. Similarly, the remaining indicator functions can be defined. By simple inspection of the payoffs in Table C.3, it can be seen that an informed buyer at $t = 1$ never chooses NT because it is dominated by a MO.

Notice also that the expected profits of a BDO submitted by an informed buyer at $t = 1$ may be positive, and as a result the informed may choose to place a BDO at $t = 1$ depending on how high the execution probability θ_1^I is null. However, the payoff at $t = 1$ of the BDO for the uninformed trader is always null (see Internet Appendix I). Therefore, we have that $\Gamma_3 = 0$, and hence $B_1 = \varnothing$ implies either Ω_3 or Γ_0 is not null. Thus,

$$P_I = p_{BLO,2}^{IH}(\mathcal{B}_1 = \varnothing) = p_{SLO,2}^{IL}(\mathcal{B}_1 = \varnothing) = 0.$$

Consequently, informed traders never choose a LO at $t = 2$, since this order is dominated by a MO. Uninformed traders also do not select a LO at $t = 2$. To see this, note that Table C.2 shows that we have to prove the result when prices do not change. In such a case we distinguish two cases: $\Omega_3 = 1$ and $\Omega_3 = 0$. In the first case, $Z = 1$ and, therefore, the expected profits of a LO are negative, as shown in Table C.2. In the second case, $B_1 = \varnothing$ due to $\Gamma_0 = 1$. Hence,

$$P_U = p_{BLO,2}^{UB}(\mathcal{B}_1 = \varnothing) = p_{SLO,2}^{US}(\mathcal{B}_1 = \varnothing) = 0,$$

and therefore, at $t = 2$ the expected profits of a LO for an uninformed trader are always null.

Let us determine next the optimal strategy for each rational trader at $t = 2$. Depending on the values of the parameters, we have 6 possible cases for the informed trader and 16 cases for the uninformed trader. Due to limits of the length of the paper, the optimal responses of uninformed traders at $t = 2$ will be specified in each equilibria (see proof of Lemma C.1), with

$$\theta_X \equiv \frac{X\kappa - k_2}{X\kappa - \frac{k_2 - k_1}{2}}, \text{ and } \theta_Y \equiv \frac{Y\kappa - k_1}{Y\kappa - \frac{1}{2}}.$$

Next, we focus on informed traders. Given that $\kappa > k_2 > k_1 \geq 1$, the following inequalities hold:

$$\frac{\kappa - k_2}{\kappa - \frac{k_2 - k_1}{2}} < \frac{\kappa - k_1}{\kappa + \frac{k_2 - k_1}{2}} < \frac{\kappa - k_1}{\kappa} < \frac{\kappa - k_1}{\kappa - \frac{1}{2}} < \frac{\kappa - k_1 + 1}{\kappa + \frac{1}{2}}.$$

Hence, the optimal strategies of the informed traders at $t = 2$ are given in Table C.5.

Condition	Optimal Strategies of Informed Traders at t = 2		
	State of the Book	IH	IL
Case I_1 $\theta_2^I \leq \frac{\kappa-k_2}{\kappa-\frac{k_2-k_1}{2}}$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	BX BMO BMO BMO BMO	SX SMO SMO SMO SMO
Case I_2 $\frac{\kappa-k_2}{\kappa-\frac{k_2-k_1}{2}} < \theta_2^I \leq \frac{\kappa-k_1}{\kappa+\frac{k_2-k_1}{2}}$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	BX BDO BMO BMO BMO	SX SMO SMO SDO SMO
Case I_3 $\frac{\kappa-k_1}{\kappa+\frac{k_2-k_1}{2}} < \theta_2^I \leq \frac{\kappa-k_1}{\kappa}$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	BX BDO BMO BDO BMO	SX SDO SMO SDO SMO
Case I_4 $\frac{\kappa-k_1}{\kappa} < \theta_2^I \leq \frac{\kappa-k_1}{\kappa-\frac{1}{2}}$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	BY BDO BMO BDO BMO	SY SDO SMO SDO SMO
Case I_5 $\frac{\kappa-k_1}{\kappa-\frac{1}{2}} < \theta_2^I \leq \frac{\kappa-k_1+1}{\kappa+\frac{1}{2}}$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	BY BDO BDO BDO BMO	SY SDO SMO SDO SDO
Case I_6 $\frac{\kappa-k_1+1}{\kappa+\frac{1}{2}} < \theta_2^I$	(A_1^1, B_1^1) (A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	BY BDO BDO BDO BDO	SY SDO SDO SDO SDO

Table C.5: Optimal Strategies of Informed Traders at $t = 2$

We define by

$$BX = \begin{cases} BMO \text{ if } p_{BLO,2}^{IH,\mathcal{B}_1=\varnothing} \leq \frac{\kappa-k_1}{\delta(\kappa+k_1-1)}, \\ BLO \text{ if } p_{BLO,2}^{IH,\mathcal{B}_1=\varnothing} > \frac{\kappa-k_1}{\delta(\kappa+k_1-1)}. \end{cases}$$

$$SX = \begin{cases} SMO \text{ if } p_{SLO,2}^{IL,\mathcal{B}_1=\varnothing} \leq \frac{\kappa-k_1}{\delta(\kappa+k_1-1)}, \\ SLO \text{ if } p_{SLO,2}^{IL,\mathcal{B}_1=\varnothing} > \frac{\kappa-k_1}{\delta(\kappa+k_1-1)}. \end{cases}$$

and

$$BY = \begin{cases} BDO \text{ if } p_{BLO,2}^{IH,\mathcal{B}_1=\varnothing} < \frac{\theta_2^I \kappa}{\delta(\kappa+k_1-1)}, \\ BLO \text{ if } p_{BLO,2}^{IH,\mathcal{B}_1=\varnothing} \geq \frac{\theta_2^I \kappa}{\delta(\kappa+k_1-1)}. \end{cases}$$

$$SY = \begin{cases} SDO \text{ if } p_{BLO,2}^{IH,\mathcal{B}_1=\varnothing} < \frac{\theta_2^I \kappa}{\delta(\kappa+k_1-1)}, \\ SLO \text{ if } p_{BLO,2}^{IH,\mathcal{B}_1=\varnothing} \geq \frac{\theta_2^I \kappa}{\delta(\kappa+k_1-1)}. \end{cases}$$

■

We next include a definition and a lemma which will be useful to prove Proposition 2.

Definition C.3 *Let us consider the following cut-off definitions*

$$\begin{aligned}
\widehat{\theta}_{MO-DO} &\equiv \frac{\kappa - k_1 - \delta\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right)}{\kappa - \delta\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right)}, \\
\overline{\theta}_{MO-DO} &\equiv \frac{\kappa - k_1 - \delta\left(\kappa - k_1 - (k_2-k_1)\left(\lambda\pi + \frac{1-\lambda}{2}\right)\right)}{\kappa - \delta\left(\kappa - k_1 - (k_2-k_1)\left(\lambda\pi + \frac{1-\lambda}{2}\right)\right)}, \qquad \text{(C.1)} \\
\widehat{\theta}_{LO-DO} &\equiv \frac{\delta\frac{1-\lambda}{2}(\kappa + k_1 - 1) - \delta\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right)}{\kappa - \delta\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right)}, \\
\underline{\theta}_{LO-DO} &\equiv \frac{\delta\frac{1-\lambda}{2}(\kappa + k_1 - 1) - \delta\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2-k_1)\frac{1-\lambda}{2}\right)}{\kappa - \delta\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2-k_1)\frac{1-\lambda}{2}\right)},
\end{aligned}$$

$$\begin{aligned}
\overline{\theta}_{LO-DO} &\equiv \frac{\delta\frac{1-\lambda}{2}(\kappa + k_1 - 1) - \delta\left(\kappa - k_1 - (k_2-k_1)\left(\lambda\pi + \frac{1-\lambda}{2}\right)\right)}{\kappa - \delta\left(\kappa - k_1 - (k_2-k_1)\left(\lambda\pi + \frac{1-\lambda}{2}\right)\right)}, \\
\widetilde{\theta}_{LO-DO} &\equiv \frac{\delta\frac{1-\lambda}{2}(\kappa + k_1 - 1) - \delta\left(\kappa - k_1 - (k_2-k_1)\frac{1-\lambda}{2}\right)}{\kappa - \delta\left(\kappa - k_1 - (k_2-k_1)\frac{1-\lambda}{2}\right)}, \\
\underline{\theta} &\equiv \frac{\kappa - k_1}{\kappa}, \text{ and} \\
\overline{\theta} &\equiv \frac{\kappa - k_1 + 1}{\kappa + \frac{1}{2}}.
\end{aligned}$$

The cutoffs defined in (C.1) satisfy the following relationships:

$$\begin{aligned}
\widehat{\theta}_{MO-DO} &\leq \overline{\theta}_{MO-DO} < \underline{\theta} \text{ and} \\
\underline{\theta}_{LO-DO} &\leq \min\left\{\widehat{\theta}_{LO-DO}, \widetilde{\theta}_{LO-DO}\right\} \leq \max\left\{\widehat{\theta}_{LO-DO}, \widetilde{\theta}_{LO-DO}\right\} \leq \overline{\theta}_{LO-DO}.
\end{aligned}$$

Lemma C.1 ***Case A.*** *Suppose $k_1 > 1$. Then, a PBE of the game is as follows:*

- $\mathcal{E}_1^D$: (BMO, SMO, BLO, SLO) *is the optimal strategy profile at $t = 1$ if*

$$\kappa_{MO-LO}^I \tau \leq \sigma, \; PIN < \psi_{LO-NT}^U \text{ and } \theta_1^I \leq \widehat{\theta}_{MO-DO}.$$

The beliefs of an uninformed trader at $t = 2$ are: $X^{1,D} = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$, $Y^{1,D} = 0$ and $Z^{1,D} = z \in [0,1]$. The optimal choice of an uninformed and an informed trader $t = 2$ are described in Table C.7 and a subset of Table C.5, respectively.[32]

[32]In the proof of the lemma, we describe for each equilibrium the relevant subset of Table C.5.

- $\mathcal{E}_2^D$: (BMO, SMO, NT, NT) *is the optimal strategy profile at* $t = 1$ *if*

$$\kappa^I_{MO-LO}\tau \leq \sigma, \ PIN \geq \psi^U_{LO-NT}, \ \text{and } \theta^I_1 \leq \overline{\theta}_{MO-DO}.$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{2,D} = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$, $Y^{2,D} = p \in [0,1]$ *and* $Z^{2,D} = z \in [0,1]$. *The optimal choice of an uninformed and an informed trader at* $t = 2$ *are described in Table C.8 and a subset of Table C.5, respectively.*

- $\mathcal{E}_3^D$: (BLO, SLO, BLO, SLO) *is the optimal strategy profile at* $t = 1$ *if*

$$\begin{aligned} & \sigma < \kappa^I_{MO-LO}\tau, \ PIN < \psi^U_{LO-NT}, \ \text{and } \theta^I_1 \leq \min\{\underline{\theta}, \widehat{\theta}_{LO-DO}\}, \\ \text{or} \quad & PIN < \psi^U_{LO-NT} \ \text{and } \underline{\theta} < \theta^I_1 \leq \min\{\overline{\theta}, \underline{\theta}_{LO-DO}\}, \\ \text{or} \quad & \overline{\theta} < \theta^I_1 \leq \underline{\theta}_{LO-DO}. \end{aligned}$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{3,D} = 0$, $Y^{3,D} = \pi$ *and* $Z^{3,D} = z \in [0,1]$. *The optimal choice of an uninformed and an informed trader* $t = 2$ *are described in Table C.9 and a subset of Table C.5, respectively.*

- $\mathcal{E}_4^D$: (BLO, SLO, NT, NT) *is the optimal strategy profile of a trader at* $t = 1$ *if*

$$\begin{aligned} & \sigma < \kappa^I_{MO-LO}\tau, \ PIN \geq \psi^U_{LO-NT}, \ \text{and } \theta^I_1 \leq \min\{\underline{\theta}, \overline{\theta}_{LO-DO}\}, \\ \text{or} \quad & PIN \geq \psi^U_{LO-NT} \ \text{and } \underline{\theta} < \theta^I_1 \leq \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\}. \end{aligned}$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{4,D} = 0$, $Y^{4,D} = 1$ *and* $Z^{4,D} = z \in [0,1]$. *The optimal choice of an uninformed and an informed trader at* $t = 2$ *are described in Table C.10 and a subset of Table C.5, respectively.*

- $\mathcal{E}_5^D$: (BDO, SDO, BLO, SLO) *is the optimal strategy profile of a trader at* $t = 1$ *if*

$$\begin{aligned} & PIN < \psi^U_{LO-NT}, \ \theta^I_1 > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\}, \ \text{and } \theta^I_2 \leq \underline{\theta}, \\ \text{or} \quad & PIN < \psi^U_{LO-NT}, \ \widetilde{\theta}_{LO-DO} < \theta^I_1, \ \text{and } \underline{\theta} < \theta^I_2 \leq \overline{\theta}, \\ \text{or} \quad & \widetilde{\theta}_{LO-DO} < \theta^I_1, \ \text{and } \overline{\theta} < \theta^I_2. \end{aligned}$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{5,D} = 0$, $Y^{5,D} = 0$ *and* $Z^{5,D} = 1$. *The optimal choice of an uninformed and an informed trader* $t = 2$ *are described in Table C.11 and a subset of Table C.5, respectively.*

- $\mathcal{E}_6^D$: (BDO, SDO, NT, NT) *is the optimal strategy profile of a trader at* $t = 1$ *if*

$$\begin{aligned} & PIN \geq \psi^U_{LO-NT}, \ \theta^I_1 > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\}, \ \text{and } \theta^I_2 \leq \underline{\theta}, \\ \text{or} \quad & PIN \geq \psi^U_{LO-NT}, \ \widetilde{\theta}_{LO-DO} < \theta^I_1, \ \text{and } \ \underline{\theta} < \theta^I_2 \leq \overline{\theta}. \end{aligned}$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{6,D} = 0$, $Y^{6,D} = p \in [0,1]$ *and* $Z^{6,D} = 1$. *The optimal choice of an uninformed and an informed trader at* $t = 2$ *are described in Table C.12 and a subset of Table C.5, respectively.*

Case B. *Suppose* $k_1 = 1$. *Then,* (BMO, SMO, NT, NT) *is the optimal strategy profile at* $t = 1$ *if* $\theta^I_1 \leq \overline{\theta}_{MO-DO}$, *and* (BDO, SDO, NT, NT) *is the optimal strategy profile at* $t = 1$ *if*

$$\begin{aligned} & \theta^I_1 > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} \ \text{and } \theta^I_2 \leq \min\left\{\underline{\theta}, \theta^I_1\right\}, \\ \text{or} \quad & \widetilde{\theta}_{LO-DO} < \theta^I_1 \ \text{and } \underline{\theta} < \theta^I_2 \leq \theta^I_1. \end{aligned}$$

Remark C.1 *Recall that in a Perfect Bayesian Equilibrium beliefs must satisfy Bayes' rule, whenever possible. This occurs along the equilibrium path, not off-the-equilibrium path, where beliefs are indeterminate. This indeterminacy might result in multiplicity of equilibria in sequential games with imperfect information. Note that this may occur in our case when the uninformed trader's beliefs at* $t = 2$ *(i.e.,* X, Y, *or* Z*) are indeterminate.*

Proof of Lemma C.1. Because of the symmetry of the model, without any loss of generality, we focus on buyers. We present the full proof for one of the possible strategy profile at $t = 1$ that yields an equilibrium. The proofs of all the other 5 equilibria can be found in the Internet Appendix II. Note that in all equilibria the optimal responses of informed traders at $t = 2$ are given in Table C.5. However, in some equilibria not all the 6 cases $I_1 - I_6$ are possible and also not all of the 5 states of the book are possible. As a result only a subset of Table C.5 will apply.

$\mathcal{E}_1^D$: (BMO, SMO, BLO, SLO)

First step. In this case $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = 0$, $\Omega_3 = 0$, $\Gamma_0 = 0$, $\Gamma_1 = 0$, $\Gamma_2 = 1$, and $\Gamma_3 = 0$. Moreover, $\theta_2^I = \theta_1^I$ and $\theta_2^U = \theta_1^U$. We define by $P \equiv p_{BLO,2}^{UB,\mathcal{B}_1=\varnothing} = p_{SLO,2}^{US,\mathcal{B}_1=\varnothing}$.

Second step. Using Bayes' rule

$$\begin{aligned} X^{1,D} &= \frac{\lambda\pi}{1-\lambda+\lambda\pi},\ Y^{1,D} = 0,\ Z^{1,D} = z \in [0,1], \\ p_{BLO,2}^{UB,\mathcal{B}_1=\varnothing} &= p_{SLO,2}^{US,\mathcal{B}_1=\varnothing} \in [0,1],\ \text{and } p_{BLO,2}^{IH,\mathcal{B}_1=\varnothing} = p_{SLO,2}^{IL,\mathcal{B}_1=\varnothing} \in [0,1]. \end{aligned}$$

Third step. Using step 2 and taking into account that $p_{BLO,2}^{UB}(\mathcal{B}_1 = \varnothing) = p_{SLO,2}^{US}(\mathcal{B}_1 = \varnothing) \in [0,1]$, at $t = 2$ the expected profits of uninformed buyers are as given by Table C.2. Using the symmetry of buyers and sellers, we obtain that the optimal strategy for the uninformed are:

Optimal Strategies of Uninformed Traders at $t = 2$		
State of the Book	**UB**	**US**
(A_1^1, B_1^1)	NT if $P = 0$ or $Z^{1,D}\kappa \geq k_1 - 1$ BLO if $P > 0$ and $Z^{1,D}\kappa < k_1 - 1$	NT if $P = 0$ or $Z^{1,D}\kappa \geq k_1 - 1$ SLO if $P > 0$ and $Z^{1,D}\kappa < k_1 - 1$
(A_1^2, B_1^1)	NT if $X^{1,D}\kappa \leq \frac{k_2-k_1}{2}$ BDO if $\frac{k_2-k_1}{2} < X^{1,D}\kappa \leq k_2$ BDO if $k_2 < X^{1,D}\kappa$ and $\theta_2^U > \theta_{X^{1,D}}$ BMO if $k_2 < X^{1,D}\kappa$ and $\theta_2^U \leq \theta_{X^{1,D}}$	SDO if $X^{1,D}\kappa < \frac{k_2-k_1}{2}$ NT if $\frac{k_2-k_1}{2} \leq X^{1,D}\kappa$
$(A_1^1, B_1^1 + \tau)$	NT	SDO
(A_1^1, B_1^2)	BDO if $X^{1,D}\kappa < \frac{k_2-k_1}{2}$ NT if $\frac{k_2-k_1}{2} \leq X^{1,D}\kappa$	NT if $X^{1,D}\kappa \leq \frac{k_2-k_1}{2}$ SDO if $\frac{k_2-k_1}{2} < X^{1,D}\kappa \leq k_2$ SDO if $k_2 < X^{1,D}\kappa$ and $\theta_2^U > \theta_{X^{1,D}}$ SMO if $k_2 < X^{1,D}\kappa$ and $\theta_2^U \leq \theta_{X^{1,D}}$
$(A_1^1 - \tau, B_1^1)$	BDO	NT

Table C.6: Optimal strategies of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

Concerning the informed buyers their expected profits are as given by Table C.1 and the optimal strategy for an informed trader at $t = 2$ is given by Table C.5.

Fourth step. Given the optimal response of traders at $t = 2$, we find the optimal action for the traders at $t = 1$ in each of the 6 cases. However, given the nature of this particular equilibrium, we can group cases and analyze them in the following way:

Case $I_1 + I_2 + I_3 : \theta_2^I \leq \dfrac{\kappa - k_1}{\kappa}$

- *Informed traders*

  As $\theta_2^I \leq \dfrac{\kappa - k_1}{\kappa}$, informed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$\begin{aligned}\kappa - k_1 &\geq \frac{1-\lambda}{2}\delta\left(\kappa + k_1 - 1\right) \text{ and} \qquad \text{(C.2)}\\ \kappa - k_1 &\geq \theta_1^I\kappa + (1-\theta_1^I)\delta\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right).\end{aligned}$$

- *Uninformed traders*

As $\theta_2^I \leq \dfrac{\kappa - k_1}{\kappa} \leq \dfrac{\kappa - k_1 + 1}{\kappa + \frac{1}{2}}$, uninformed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$\left(\lambda\pi + 1 - \lambda\right)\left(k_1 - 1\right) - \lambda\pi\kappa > 0. \qquad \text{(C.3)}$$

Case $I_4 + I_5 + I_6 : \dfrac{\kappa - k_1}{\kappa} < \theta_2^I$

- *Informed traders*

Consider an informed buyer at $t = 1$. If he chooses a BMO, then he obtains

$$\mathbb{E}\left(\Pi_{BMO,1}^{IH}\right) = \left(\kappa - k_1\right)\tau.$$

If instead he deviates towards a BDO, he will obtain

$$\mathbb{E}\left(\Pi_{BDO,1}^{IH}\right) = \theta_1^I\kappa\tau + (1-\theta_1^I)\delta\left[\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} + (\kappa - k_1) - (k_2 - k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right]\tau.$$

Combining the previous expression and the fact that $\dfrac{\kappa - k_1}{\kappa} < \theta_2^I = \theta_1^I$, it follows that

$$\mathbb{E}\left(\Pi_{BDO,1}^{IH}\right) > \mathbb{E}\left(\Pi_{BMO,1}^{IH}\right) \qquad \text{(C.4)}$$

is always satisfied, and we conclude that in this case there is no equilibrium in which the strategy profile chosen at $t = 1$ is (BMO, SMO, BLO, SLO).

Fifth step. Based on the above, nobody at $t = 1$ has unilateral incentives to deviate whenever $\theta_1^I \leq \dfrac{\kappa - k_1}{\kappa}$ and (C.2), (C.3) and (C.4) are satisfied. These conditions can be rewritten as

$$\kappa_{MO-LO}^I\tau \leq \sigma,\ PIN < \psi_{LO-NT}^U \text{ and } \theta_1^I \leq \widehat{\theta}_{MO-DO}. \qquad \text{(C.5)}$$

Finally, in the following tables we include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BMO, SMO, BLO, SLO) is the strategy profile chosen at $t = 1$ and the fact that in this case $\theta_2^I = \theta_1^I$. In relation to informed traders the optimal choice at $t = 2$ is obtained by selecting in Table C.5 the cases I_1, I_2 and I_3 and the following possible prices (A_1^2, B_1^1), $(A_1^1, B_1^1 + \tau)$, (A_1^1, B_1^2), $(A_1^1 - \tau, B_1^1)$.

Concerning uninformed traders notice that the condition $\left(\lambda\pi + 1 - \lambda\right)\left(k_1 - 1\right) - \lambda\pi\kappa > 0$ implies that $X^{1,D}\kappa < k_1 - 1 < k_2$. Hence, the optimal choice of uninformed traders at $t = 2$ is as follows:

Condition	Optimal Choice of Uninformed Traders at t = 2		
	State of the Book	**UB**	**US**
Case $U_1^{\mathcal{E}_1^D}$ $k_1 - 1 \le \frac{k_2-k_1}{2}$ or $k_1 - 1 > \frac{k_2-k_1}{2}$ and $X^{1,D}\kappa < \frac{k_2-k_1}{2}$	(A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT NT BDO BDO	SDO SDO NT NT
Case $U_2^{\mathcal{E}_1^D}$ $k_1 - 1 > \frac{k_2-k_1}{2}$ and $X^{1,D}\kappa = \frac{k_2-k_1}{2}$	(A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT NT NT BDO	NT SDO NT NT
Case $U_3^{\mathcal{E}_1^D}$ $k_1 - 1 > \frac{k_2-k_1}{2}$ and $\frac{k_2-k_1}{2} < X^{1,D}\kappa < k_1 - 1$	(A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	BDO NT NT BDO	NT SDO SDO NT

Table C.7: Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO)

$\mathcal{E}_2^D$: (BMO, SMO, NT, NT)

In this case the beliefs of an uninformed trader at $t = 2$ are: $X^{2,D} = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$, $Y^{2,D} = p \in [0,1]$ and $Z^{2,D} = z \in [0,1]$. In addition, nobody at $t = 1$ has unilateral incentives to deviate whenever

$$\kappa^I_{MO-LO}\tau \le \sigma,\ PIN \ge \psi^U_{LO-NT},\ \text{and}\ \theta_1^I \le \overline{\theta}_{MO-DO}. \tag{C.6}$$

Finally, we include the decisions that are in the equilibrium path and that in this case $\theta_2^I = \theta_1^I$. In relation to uninformed traders, and taking into account that $k_1 - 1 \le X^{2,D}\kappa$, we obtain

Condition	Optimal Choice of Uninformed Traders at t = 2		
	State of the Book	**UB**	**US**
Case $U_1^{\mathcal{E}_2^D}$ $k_1 - 1 \le X^{2,D}\kappa < \frac{k_2-k_1}{2}$	(A_1^1, B_1^1) (A_1^2, B_1^1) (A_1^1, B_1^2)	NT NT BDO	NT SDO NT
Case $U_2^{\mathcal{E}_2^D}$ $k_1 - 1 < X^{2,D}\kappa = \frac{k_2-k_1}{2}$	(A_1^1, B_1^1) (A_1^2, B_1^1) (A_1^1, B_1^2)	NT NT NT	NT NT NT
Case $U_3^{\mathcal{E}_2^D}$ $\max\left\{k_1 - 1, \frac{k_2-k_1}{2}\right\} < X^{2,D}\kappa \le k_2$ or $k_2 < X^{2,D}\kappa$ and $\theta_2^U > \theta_{X^{2,D}}$	(A_1^1, B_1^1) (A_1^2, B_1^1) (A_1^1, B_1^2)	NT BDO NT	NT NT SDO
Case $U_4^{\mathcal{E}_2^D}$ $k_2 < X^{2,D}\kappa$ and $\theta_2^U \le \theta_{X^{2,D}}$	(A_1^1, B_1^1) (A_1^2, B_1^1) (A_1^1, B_1^2)	NT BMO NT	NT NT SMO

Table C.8: Optimal choice of uniformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, NT, NT)

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table C.5 only the cases I_1, I_2 and I_3, and the following possible prices: $(A_1^1, B_1^1), (A_1^2, B_1^1)$, and (A_1^1, B_1^2), with $BX = BMO$, $SX = SMO$, $BY = BDO$, and $SY = SDO$.

$\mathcal{E}_3^D$: (BLO, SLO, BLO, SLO)
In this case the beliefs of an uninformed trader at $t = 2$ are: $X^{3,D} = 0$, $Y^{3,D} = \pi$ and $Z^{3,D} = z \in [0, 1]$. In addition, the conditions under which nobody is willing to deviate at $t = 1$ are

$$\begin{array}{ll} & \sigma < \kappa_{MO-LO}^I \tau,\ PIN < \psi_{LO-NT}^U, \text{ and } \theta_1^I \le \min\{\underline{\theta}, \widehat{\theta}_{LO-DO}\}, \\ \text{or} & PIN < \psi_{LO-NT}^U \text{ and } \underline{\theta} < \theta_1^I \le \min\{\overline{\theta}, \underline{\theta}_{LO-DO}\}, \\ \text{or} & k_1 > 1 \text{ and } \overline{\theta} < \theta_1^I \le \underline{\theta}_{LO-DO}. \end{array} \tag{C.7}$$

Finally, we include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BLO, SLO, BLO, SLO) is the strategy profile chosen at $t = 1$ and $\theta_2^I = \theta_1^I$.
Concerning uninformed traders, it follows that their optimal choices at $t = 2$ are

Condition	**Optimal Choice of Uninformed Traders at t = 2**		
	State of the Book	**UB**	**US**
Case $U_1^{\mathcal{E}_3^D}$ $Y^{3,D}\kappa \le \frac{1}{2}$	(A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT NT BDO BDO	SDO SDO NT NT
Case $U_2^{\mathcal{E}_3^D}$ $\frac{1}{2} < Y^{3,D}\kappa \le k_1$ or $Y^{3,D}\kappa > k_1$and $\theta_2^U > \theta_{Y^{3,D}}$	(A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT BDO BDO NT	SDO NT NT SDO
Case $U_3^{\mathcal{E}_3^D}$ $Y^{3,D}\kappa > k_1$ and $\theta_2^U \le \theta_{Y^{3,D}}$	(A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT BMO BDO NT	SDO NT NT SMO

Table C.9: Optimal choice of uninformed traders when the strategy profile at $t = 1$ is (BLO, SLO, BLO, SLO).

In relation to informed traders the optimal choice at $t = 2$ is obtained by selecting in Table C.5 all the cases I_1- I_6 and the prices: (A_1^2, B_1^1), $(A_1^1, B_1^1 + \tau)$, (A_1^1, B_1^2), and $(A_1^1 - \tau, B_1^1)$.

$\mathcal{E}_4^D$: (BLO, SLO, NT, NT)
The beliefs of an uninformed trader at $t = 2$ are: $X^{4,D} = 0$, $Y^{4,D} = 1$ and $Z^{4,D} = z \in [0, 1]$. In addition, the conditions under which nobody is willing to deviate at $t = 1$ are

$$\begin{array}{ll} & \sigma < \kappa_{MO-LO}^I \tau,\ PIN \ge \psi_{LO-NT}^U, \text{ and } \theta_1^I \le \min\{\underline{\theta}, \overline{\theta}_{LO-DO}\}, \\ \text{or} & PIN \ge \psi_{LO-NT}^U \text{ and } \underline{\theta} < \theta_1^I \le \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\}. \end{array} \tag{C.8}$$

We include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BLO, SLO, NT, NT) is the strategy profile chosen at $t = 1$ and the fact that in this case $\theta_2^I = \theta_1^I$ and $\theta_2^U = \theta_1^U$. Concerning the uninformed traders, we have

Condition	Optimal Choice of Uninformed Traders at t = 2		
	State of the Book	**UB**	**US**
Case $U_1^{\mathcal{E}_4^D}$ $\theta_2^U > \theta_{Y^{4,D}}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1+\tau)$	BMO	NT
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1-\tau, B_1^1)$	NT	SMO
Case $U_2^{\mathcal{E}_4^D}$ $\theta_2^U \leq \theta_{Y^{4,D}}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1+\tau)$	BDO	NT
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1-\tau, B_1^1)$	NT	SDO

Table C.10: Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, NT, NT).

In relation to informed traders the optimal choice at $t = 2$ is obtained by selecting in Table C.5 all the cases I_1- I_5, for all the possible pairs of best prices, with $BX = BMO$, $SX = SMO$, $BY = BDO$, and $SY = SDO$.

$\mathcal{E}_5^D$: (BDO, SDO, BLO, SLO)

In this case the beliefs of an uninformed trader at $t = 2$ are: $X^{5,D} = 0$, $Y^{5,D} = 0$ and $Z^{5,D} = 1$. In addition, nobody at $t = 1$ has unilateral incentives to deviate from (BDO, SDO, BLO, SLO) whenever

$$\begin{aligned} & PIN < \psi^U_{LO-NT},\ \theta_1^I > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\},\text{ and } \theta_2^I \leq \underline{\theta}, \\ \text{or}\quad & PIN < \psi^U_{LO-NT},\ \widetilde{\theta}_{LO-DO} < \theta_1^I,\text{ and } \underline{\theta} < \theta_2^I \leq \overline{\theta}, \\ \text{or}\quad & k_1 > 1,\ \widetilde{\theta}_{LO-DO} < \theta_1^I,\text{ and } \overline{\theta} < \theta_2^I. \end{aligned} \tag{C.9}$$

Notice that in this equilibrium we always have $\theta_2^I < \theta_1^I$. Furthermore, the optimal responses of uninformed traders are

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
$(A_1^1, B_1^1+\tau)$	NT	SDO
(A_1^1, B_1^2)	BDO	NT
$(A_1^1-\tau, B_1^1)$	BDO	NT

Table C.11: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BDO, SDO, BLO, SLO).

In relation to informed traders the optimal choice at $t = 2$ is obtained by selecting in Table C.5 all the cases I_1- I_6, for all the possible pairs of best prices, with $BX = BMO$, $SX = SMO$, $BY = BDO$, and $SY = SDO$.

$\mathcal{E}_6^D$: (BDO, SDO, NT, NT)

In this case the beliefs of an uninformed trader at $t = 2$ are: $X^{6,D} = 0$, $Y^{6,D} = p \in [0, 1]$ and $Z^{6,D} = 1$. In addition, nobody at $t = 1$ has unilateral incentives to deviate from (BDO, SDO, NT, NT) whenever

$$\begin{aligned} & PIN \geq \psi^U_{LO-NT},\ \theta_1^I > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\},\text{ and } \theta_2^I \leq \underline{\theta}, \\ \text{or}\quad & PIN \geq \psi^U_{LO-NT},\ \widetilde{\theta}_{LO-DO} < \theta_1^I,\text{ and } \underline{\theta} < \theta_2^I \leq \overline{\theta}, \\ \text{or}\quad & k_1 = 1,\ \widetilde{\theta}_{LO-DO} < \theta_1^I \text{ and } \overline{\theta} < \theta_2^I \leq \theta_1^I. \end{aligned} \tag{C.10}$$

Notice that in this equilibrium we also have that $\theta_2^I \leq \theta_1^I$. Furthermore, the optimal responses of uninformed traders are

Optimal Choice of Uninformed Traders at $t = 2$		
State of the Book	**UB**	**US**
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
(A_1^1, B_1^2)	BDO	NT

Table C.12: Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BDO, SDO, NT, NT).

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table C.5 all the cases $I_1 - I_6$ and the following possible prices: (A_1^1, B_1^1), (A_1^2, B_1^1), (A_1^1, B_1^2), with $BX = BMO$, $SX = SMO$, $BY = BDO$, and $SY = SDO$.

Finally, note that substituting $k_1 = 1$ (Case B) into the expressions of κ^I_{MO-LO} and ψ^U_{LO-NT}, we have that

$$\kappa^I_{MO-LO} = \frac{1}{1 - \frac{1}{2}\delta(1-\lambda)} \text{ and } \psi^U_{LO-NT} = 0.$$

Moreover, since $\kappa^I_{MO-LO} < 2$, it follows that $\kappa^I_{MO-LO}\tau < \sigma$ and $PIN \geq \psi^U_{LO-NT}$. Therefore, using (C.5)-(C.10), we have that when $k_1 = 1$, the conditions related to $\mathcal{E}_1^D$, $\mathcal{E}_3^D$ and $\mathcal{E}_5^D$ are not satisfied. Moreover, as in this case the expected profits of a MO are higher than those of a LO for an informed trader, $\mathcal{E}_4^D$ cannot be an equilibrium when $k_1 = 1$. Thus, in this case we have two possible equilibria: $\mathcal{E}_2^D$ and $\mathcal{E}_6^D$. Specifically, (BMO, SMO, NT, NT) is the optimal strategy profile at $t = 1$ if $\theta_1^I \leq \overline{\theta}_{MO-DO}$, and (BDO, SDO, NT, NT) is the optimal strategy profile at $t = 1$ if

$$\begin{aligned} &\theta_1^I > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} \text{ and } \theta_2^I \leq \min\{\underline{\theta}, \theta_1^I\}, \\ \text{or } \quad &\widetilde{\theta}_{LO-DO} < \theta_1^I \text{ and } \underline{\theta} < \theta_2^I \leq \theta_1^I. \end{aligned}$$

■

Proof of Proposition 2. **Case A.** We consider the same four possible cases, depending on the initial conditions of the single-venue market.

Case A.1: $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$

We start with market conditions such that the prevailing equilibrium is $\mathcal{E}_3^{ND}$, where conditions (B.7) and (B.8) are satisfied. When there is access to the DP, out of the 6 equilibria, there are only two possible equilibria that satisfy these conditions: $\mathcal{E}_3^D$ and $\mathcal{E}_5^D$. From Lemma C.1 we can see that $\mathcal{E}_3^D$ is an equilibrium if conditions (C.7) are satisfied. Using the relationship between the cutoffs in this case (see Internet Appendix II for a full proof), we conclude that $\mathcal{E}_3^D$ is an equilibrium whenever

$$\begin{aligned} &\theta_1^I \leq \underline{\theta}_{LO-DO} \quad \text{if } \underline{\theta} < \widehat{\theta}_{LO-DO}, \\ \text{or } \quad &\theta_1^I \leq \widehat{\theta}_{LO-DO} \quad \text{if } \widehat{\theta}_{LO-DO} \leq \underline{\theta}. \end{aligned}$$

On the other hand, $\mathcal{E}_5^D$ is an equilibrium if conditions (C.9) are satisfied, they can be rewritten as

$$\begin{aligned} &\theta_1^I > \overline{\theta}_{LO-DO} \text{ and } \theta_2^I \leq \underline{\theta}, \\ \text{or } \quad &\widetilde{\theta}_{LO-DO} < \theta_1^I \text{ and } \underline{\theta} < \theta_2^I. \end{aligned}$$

As a result, when $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$, the optimal strategy profiles at $t = 1$ are

$$\begin{cases} (BLO, SLO, BLO, SLO) & \text{if } \theta_1^I \leq \theta^1_{LO-LO}, \\ (BDO, SDO, BLO, BLO) & \text{if } \theta_1^I > \theta^1_{DO-LO}, \end{cases}$$

where

$$\theta^1_{LO-LO} = \begin{cases} \widehat{\theta}_{LO-DO} & \text{if} \quad \widehat{\theta}_{LO-DO} < \underline{\theta}, \\ \underline{\theta}_{LO-DO} & \text{otherwise.} \end{cases}$$

$$\theta^1_{DO-LO} = \begin{cases} \overline{\theta}_{LO-DO} & \text{if} \quad \theta^I_2 \leq \underline{\theta}, \\ \widetilde{\theta}_{LO-DO} & \text{otherwise.} \end{cases}$$

Case A.2: $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN \geq \psi^U_{LO-NT}$

We start with market conditions such that the prevailing equilibrium is $\mathcal{E}^{ND}_4$, where conditions (B.9) and (B.10) are satisfied. When there is access to the DP, out of the 6 equilibria there are only three possible equilibria that satisfy these conditions: $\mathcal{E}^D_4$, $\mathcal{E}^D_5$, and $\mathcal{E}^D_6$. From Lemma C.1 we can see that $\mathcal{E}^D_4$ is an equilibrium if conditions (C.8) are satisfied, then

$$\theta^I_1 \leq \overline{\theta}_{LO-DO}.$$

On the other hand, $\mathcal{E}^D_5$ is an equilibrium if conditions (C.9) are satisfied, they can be rewritten as

$$\widetilde{\theta}_{LO-DO} < \theta^I_1 \text{ and } \overline{\theta} < \theta^I_2.$$

Finally, $\mathcal{E}^D_6$ is an equilibrium if conditions (C.10) are satisfied, and they can be rewritten as

$$\begin{array}{rl} & \theta^I_1 > \overline{\theta}_{LO-DO} \text{ and } \theta^I_2 \leq \underline{\theta}, \\ \text{or} & \widetilde{\theta}_{LO-DO} < \theta^I_1 \text{ and } \underline{\theta} < \theta^I_2 \leq \overline{\theta}. \end{array}$$

As a result, the optimal strategy profiles of a trader at $t = 1$ are

$$\begin{cases} (BLO, SLO, NT, NT) & \text{if} \quad \theta^I_1 \leq \theta^{22}_{LO-NT}, \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta^{22}_{DO-NT} < \theta^I_1 \leq \theta^{22}_{DO-LO}, \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta^I_1 > \theta^{22}_{DO-LO}, \end{cases}$$

where

$$\theta^{22}_{LO-NT} = \begin{cases} \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\} & \text{if} \quad \underline{\theta} < \overline{\theta}_{LO-DO}, \\ \overline{\theta}_{LO-DO} & \text{otherwise,} \end{cases}$$

$$\theta^{22}_{DO-NT} = \begin{cases} \overline{\theta}_{LO-DO} & \text{if} \quad \theta^I_2 \leq \underline{\theta}, \\ \widetilde{\theta}_{LO-DO} & \text{if} \quad \underline{\theta} < \theta^I_2 \leq \overline{\theta}, \\ 1 & \text{if} \quad \overline{\theta} < \theta^I_2, \end{cases} \quad \text{and}$$

$$\theta^{22}_{DO-LO} = \begin{cases} 1 & \text{if} \quad \theta^I_2 \leq \overline{\theta}, \\ \widetilde{\theta}_{LO-DO} & \text{if} \quad \text{otherwise.} \end{cases}$$

Case A.3: $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN < \psi^U_{LO-NT}$

We start with market conditions such that the prevailing equilibrium is $\mathcal{E}^{ND}_1$, where conditions (B.3) and (B.4) are satisfied. When there is access to the DP, out of the 6 equilibria there are only two possible equilibria that satisfy these conditions: $\mathcal{E}^D_1$ and $\mathcal{E}^D_5$. From Lemma C.1 we can see that $\mathcal{E}^D_1$ is an equilibrium if conditions (C.5) are satisfied. Similarly, $\mathcal{E}^D_5$ is an equilibrium if conditions (C.9) are satisfied, they can be rewritten as

$$\begin{array}{rl} & \theta^I_1 > \overline{\theta}_{MO-DO} \text{ and } \theta^I_2 \leq \underline{\theta}, \\ \text{or} & \widetilde{\theta}_{LO-DO} < \theta^I_1 \text{ and } \underline{\theta} < \theta^I_2. \end{array}$$

As a result, the optimal strategy profiles of a trader at $t = 1$ are :

$$\begin{cases} (BMO, SMO, BLO, BLO) & \text{if} \quad \theta^I_1 \leq \widehat{\theta}_{MO-DO}, \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta^I_1 > \theta^{21}_{DO-LO}, \end{cases}$$

where

$$\theta^{21}_{DO-LO} = \begin{cases} \overline{\theta}_{MO-DO} & \text{if} \quad \theta^I_2 \leq \underline{\theta}, \\ \widetilde{\theta}_{LO-DO} & \text{otherwise.} \end{cases}$$

Case A.4: $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^U_{LO-NT}$

We start with market conditions such that the prevailing equilibrium is $\mathcal{E}^{ND}_2$, where conditions (B.5) and (B.6) are satisfied. When there is access to the DP, out of the 6 equilibria there are only three possible equilibria that satisfy these conditions: $\mathcal{E}^D_2$, $\mathcal{E}^D_5$, and $\mathcal{E}^D_6$. From Lemma C.1 we can see that $\mathcal{E}^D_2$ is an equilibrium if conditions (C.6) are satisfied. Similarly, $\mathcal{E}^D_5$ is an equilibrium if conditions (C.9) are satisfied, they can be rewritten as

$$\widetilde{\theta}_{LO-DO} < \theta^I_1 \text{ and } \overline{\theta} < \theta^I_2.$$

Finally, $\mathcal{E}^D_6$ is an equilibrium if conditions (C.10) are satisfied and in this case they can be rewritten as

$$\begin{aligned} &\theta^I_1 > \overline{\theta}_{MO-DO} \text{ and } \theta^I_2 \leq \underline{\theta}, \\ \text{or} \quad &\widetilde{\theta}_{LO-DO} < \theta^I_1 \text{ and } \underline{\theta} < \theta^I_2 \leq \overline{\theta}. \end{aligned}$$

As a result, the optimal strategy profiles of a trader at $t = 1$ are

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta^I_1 \leq \overline{\theta}_{MO-DO}, \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta^3_{DO-NT} < \theta^I_1 \leq \theta^3_{DO-LO}, \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta^I_1 > \theta^3_{DO-LO}, \end{cases}$$

where

$$\begin{aligned} \theta^3_{DO-NT} &= \begin{cases} \overline{\theta}_{MO-DO} & \text{if} \quad \theta^I_2 \leq \underline{\theta}, \\ \widetilde{\theta}_{LO-DO} & \text{if} \quad \underline{\theta} < \theta^I_2 \leq \overline{\theta}, \\ 1 & \text{if} \quad \overline{\theta} < \theta^I_2. \end{cases} \quad \text{and} \\ \theta^3_{DO-LO} &= \begin{cases} 1 & \text{if} \quad \theta^I_2 \leq \overline{\theta}, \\ \widetilde{\theta}_{LO-DO} & \text{if} \quad \overline{\theta} < \theta^I_2. \end{cases} \end{aligned}$$

Case B. From Lemma C.1, one can see that when there is access to the DP, out of the 6 equilibria there are only two possible equilibria that satisfy these conditions: $\mathcal{E}^D_2$ and $\mathcal{E}^D_6$. Note that $\mathcal{E}^D_2$ is an equilibrium if conditions (C.6) are satisfied, while $\mathcal{E}^D_6$ is an equilibrium if

$$\begin{aligned} &\theta^I_1 > \overline{\theta}_{MO-DO} \text{ and } \theta^I_2 \leq \underline{\theta}, \\ \text{or} \quad &\widetilde{\theta}_{LO-DO} < \theta^I_1 \text{ and } \underline{\theta} < \theta^I_2. \end{aligned}$$

As a result, the optimal strategy profiles of a trader at $t = 1$ are

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta^I_1 \leq \overline{\theta}_{MO-DO}, \\ (BDO, SDO, NT, NT) & \text{if} \quad \widehat{\theta}^3_{DO-NT} < \theta^I_1, \end{cases}$$

where

$$\widehat{\theta}^3_{DO-NT} = \begin{cases} \overline{\theta}_{MO-DO} & \text{if} \quad \theta^I_2 \leq \underline{\theta}, \\ \widetilde{\theta}_{LO-DO} & \text{if} \quad \theta^I_2 > \underline{\theta}. \end{cases}$$

■

Proof of Propositions 3, 4, and 5. See Internet Appendix III. ■

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