Information and Optimal Trading Strategies with Dark Pools *

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Abstract

We study the competition between two trading venues with different degrees of transparency – a transparent exchange organized as a limit order book and an opaque dark pool –in the presence of asymmetric information. We find that the optimal order submission strategies depend on stock market characteristics (volatility, liquidity, adverse selection) and traders characteristics (immediacy and information). Adding a dark pool not only enlarges traders' strategies set but also may induce a substitution of trading venue, order type, and increase market participation in relation to when the dark pool is unavailable. Consequently, dark trading affects market quality and investor's welfare. Our results reconcile the ambiguous effects of dark pools on market performance found both in the previous theoretical and empirical studies.

Keywords: trading venues, dark liquidity, limit order book, price risk, adverse selection, double volume cap

JEL codes: G12, G14, G18

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1 Introduction

In today's financial markets traders have access to competing venues for trading stocks, which differ in their level of transparency. In addition to traditional exchanges (lit markets), market participants can also trade in dark pools.[1] Dark pools grew as a result of technological innovations and changes in regulatory framework. Their current consolidated trading volume in the US equity markets is around 13.8%, while in European equity markets is around 4.57%.[2] The increase in the importance of dark pools as trading venues, and the segmentation of the order flow into lit and dark venues has led regulators to focus on the impact of dark trading on market quality and investors' welfare.[3] With the objective of promoting more efficient and transparent markets, the European Commission passed MiFID II, while SEC adopted in 2018 an amendment to regulation Alternative Trading Systems. This paper studies how the introduction of a dark pool alongside a transparent exchange affects market quality and welfare in the presence of long-lived asymmetric information. We show that the effects of a dark pool on market quality and welfare depend on stock market and trader characteristics. Our results reconcile conflicting empirical evidence of adding a dark pool alongside an exchange and have implications for the current policy debate regarding the regulation of dark pools.

Our model reflects the main characteristics of today's financial markets. The exchange is organized as a fully transparent limit order book. In addition to the exchange, we model an alternative trading venue: a dark pool. Despite the fact that there exist many types of dark pools, our modeling captures two of their main features: (1) no pre-trade transparency since dark pools are completely opaque in the sense that do not quote the liquidity that is available and this makes execution uncertain; (2) they do not determine prices and derive them from the ones prevailing in the exchange as the midpoint between the best bid and ask prices at any point in time. This type of pricing is typical of dark pools which are owned by agency brokers or exchanges and represent 57% of the consolidated dark trading volume in the US and 67% in Europe (Buti et al., 2017).[4]

We build a sequential trading model with three periods. In each of the first two trading periods, a new trader arrives to the market. There are two possible types of traders: rational or liquidity. Liquidity traders trade for liquidity reasons and only submit market orders to the exchange to ensure execution. Rational traders strategically choose whether or not to trade, and if they trade, they simultaneously select the venue and the type of order than maximizes profits given their information. Thus, if rational traders choose to trade, then they can submit a variety of order types: market order (demand liquidity) or limit order (supply liquidity) to the exchange, or a dark pool order. In addition, rational traders may be informed if they perfectly know the liquidation value of the

[1]Dark pools are "trading venues or mechanisms containing anonymous, non-displayed liquidity that is available for execution " (Banks, 2014).

[2]See Rossenblat Securities, Let there be light, February 2019, available at https://www.rblt.com.

[3]Segmentation of the order flow refers to a situation where the optimal trading strategies of informed and uninformed traders are such that they trade in different trading venues.

[4]Other types of dark pools offer other types types of price improvements which are not necessarily equal to the midpoint. See Brolley (2018) for an analysis of large and small dark pool price improvements.

asset, or (privately) uninformed if they only know the distribution of the liquidation value of the asset conditional on the information provided by the book. One of the reasons we use a sequential trading model is to understand how long-lived asymmetric information is reflected in prices and how uninformed traders learn from the limit order book.

We compare traders' optimal strategies and market quality in a model where traders have access to two trading venues (a dark pool and an exchange) to one where agents are restricted to trading in the exchange (benchmark). In the first trading period of the benchmark model, we show that when the asset's volatility is higher than a cut-off value, an informed trader has a large informational advantage and chooses to demand liquidity (because the price improvement of a limit order is not sufficient to compensate for its execution risk). Otherwise, an informed trader supplies liquidity. By contrast, an uninformed trader never chooses a market order and the choice between a limit order and not to trade depends on the degree of adverse selection he faces. An uninformed selects a limit order only when there is a high probability that his order will be executed against the order of a liquidity trader instead of an informed trader. Concerning the second trading period of the benchmark model, an informed trader is always interested in trading due to his informational advantage in relation to the other investors, whereas the choice of an uninformed trader is between demanding liquidity or not trading and this depends on the information about the value of the asset inferred by the uninformed traders compared to the best prices.

Summing up, traders' optimal strategies in the first trading period depend on stock characteristics, such as volatility and degree of information asymmetry. Combining these dimensions, we distinguish four types of stocks, depending on whether volatility and information asymmetry are high or low. This categorization in four types of stocks will be the basis of our comparison of the effects of adding a dark pool alongside an exchange affects market quality, and drives our empirical implications.

Adding a dark pool not only enlarges traders' strategies set but also may induce a substitution of trading venue, order type, and increases market participation in relation to the framework where the dark pool is unavailable. In the first trading period, informed traders find dark orders more attractive than limit and market orders when the execution risk in the dark perceived by these traders is sufficiently low. In contrast, uninformed traders do not go to the dark pool in the first trading period, since the price improvement is not sufficient to determine them to trade or change trading venue. Nevertheless, introducing the dark pool alongside the exchange may switch uninformed traders' optimal submission strategy from no trading to supplying liquidity in the exchange. This occurs when adverse selection in the exchange is reduced because informed traders migrate from the exchange to the dark. Therefore, when the informed trader migrates to the dark venue, there is segmentation of the order flow. In the second period of trading, we find that, ceteris paribus, the execution probability in the dark pool is greater for an uninformed than for an informed trader when there has been previous order migration to the dark pool. Informed and uninformed traders submit dark orders if the execution risk in the dark is low enough and the price improvement is significative. Whether there is segmentation of the order flow in the second period of trading depends on stock and trader characteristics.

The introduction of a dark pool alongside an exchange has the following effects on each indicator of market quality and investors' welfare. First, price informativeness is lower in the first trading period when the informed trader migrates to the dark pool and the uninformed remains in the exchange. In the second trading period, the effects are generally ambiguous, depending also on the segmentation of the order flow. For example, price informativeness might increase price in the second trading period when an uninformed trader submits a dark pool order and the informed trader remains in the exchange. This case recovers the main finding of Zhu (2014).

Second, for high volatility stocks the expected inside spread is always lower or equal with the introduction of the dark pool in both trading periods. However, for low volatility stocks, the spread in the first trading period is always higher or equal with the dark pool, while it is ambiguous in the second trading period. The results for low volatility stocks are aligned with the conjecture in Buti et al. (2017) that dark trading would not necessarily cause a wider spread even if asymmetric information were introduced, but their conjecture does not contemplate our different results for high volatility stocks.

Third, for high volatility stocks trading volume in the exchange and total trading volume is always lower or equal with the introduction of the dark pool in the first trading period. However, when the volatility is low, trading volume in the exchange is unchanged, while total trading volume is higher or equal with the dark pool in the first period of trading. These results are driven by the different effects that the market and limit order migration have on volume. When market orders migrate from the exchange to the dark pool then total volume decreases, whereas when limit orders migrate to the dark pool, then the total volume increases.

Fourth, in terms of investors' welfare, the expected profits of each type of trader in the first trading period are not lower with the dark pool than without it. However, in the second trading period, the profits of uninformed and liquidity traders may be reduced due to the introduction of the dark pool except if the stock's volatility is high and asymmetry of information is low.

Our paper is closely related to the theoretical literature that analyzes the effects of adding a dark pool alongside an exchange on market performance. Competition between dealer markets and other forms of exchange, such as passive crossing networks (similar to dark pools), with asymmetric information has also been analyzed by Hendershott and Mendelson (2000) and Degryse et al. (2009). Hendershott and Mendelson (2000), in a static set-up, find that a crossing network imposes positive liquidity externalities and negative crowding externalities on each other and, therefore, have ambiguous effects on the inside spread. Degryse et al. (2009) show that the same positive and negative externalities are preserved in a dynamic setup and analyze how welfare and the order flow dynamics depend on the degree of market transparency.[5]

Our work complements the previous theoretical work that studies the competition between an exchange and a dark pool. To the best of our knowledge, we are the first to model the competition

[5]Our research is also related to two other broader strands of the literature: competition between multiple trading venues (Pagano, 1989; Chowdry and Nanda, 1991) and transparency (Biais, 1993; Madhavan 1995; Frutos and Manzano, 2002; Frutos and Manzano, 2005; Dumitrescu, 2010; Boulatov and George, 2013, among others).

between an exchange that is organized as a limit order book and a dark pool in the presence of long-lived asymmetric information.[6] As in Zhu (2014) we examine the role of asymmetric information in competing trading venues. However, we model the competition of a dark pool with a limit order book and, therefore, in our framework traders can demand liquidity (as in Zhu, 2014) but also supply liquidity to the exchange. Moreover, we propose a sequential model which allows us to examine how information is incorporated in the book gradually and how this is reflected in the traders' strategies. Interestingly, we find that in the second trading period, when the informed trader stays in the exchange and the uninformed prefers to trade in the dark pool, we recover the result of Zhu (2014) that dark pools improve price informativeness. However, we show that when market conditions are such that the informed trader migrates to the dark pool and the uninformed stays in the exchange, then the dark pool harms price informativeness.[7]

Buti et al. (2017) and Brolley (2018) examine the competition between a fully transparent limit order book and a dark pool. In a symmetric information setup with private values, Buti et al. (2017) show that the introduction of a dark pool that competes with an illiquid limit order book is on average associated with trade creation, wider spread, lower depth and welfare deterioration. To complement their work, we introduce asymmetric information in a common value setup and recover some of their results for low volatility stocks (when the value of information for an informed trader is low) in the first trading period. However, since there is learning from prices in the second trading period, our market quality results differ fundamentally. Brolley (2018), using a model with asymmetric information, shows that the impact of dark trading on market quality depends on the relative price improvement of dark orders over limit orders. In contrast to Brolley (2018), who studies how different levels of dark pool price improvement affect market quality, we develop a sequential trading model with long-lived information where the dark pool reference price is the mid-point of the exchange. Our results differ from Brolley (2018) since he finds that market quality improves for a mid-point dark pool, while we show that the effects of the introduction of a dark pool on market quality depend on the market quality indicator, the trading period, and stock and trader characteristics. The differences from both trading periods emerge because in our model an uninformed trader uses the limit order book to extract information about the common value of the asset.

The fact that market quality and welfare depend on stock and trader characteristics helps us reconcile the mixed results found in the empirical literature regarding the effect of adding a dark pool on the market performance of the exchange. Thus, we show that adding a dark pool in a market where a low volatility stock is traded has an initial negative effect on liquidity (which is consistent with the empirical studies of Nimalendran and Ray, 2014; Weaver, 2014; Kwan et al., 2015; Degryse et al., 2015; Hatheway et al., 2017), while adding a dark pool in a market where a high

[6] Glosten (1994), Chakravarty and Holden (1995), Seppi (1997), Biais et al. (2000) and Kaniel and Liu (2006) emphasize the role of asymmetric information in the choice of order submission strategies in a single trading venue. In addition, Parlour (1998), Foucault (1999), Foucault, Kadan and Kandel (2005), Goettler et al. (2009), Rosu (2009) study the optimal choice of order type in dynamic models.

[7] Ye (2011) finds that adding a dark pool alongside a dealer market always reduces price informativeness if the uninformed is restricted to trading in the exchange.

volatility stock is traded has a positive effect on liquidity (Gresse, 2006; Buti et al., 2011; Ready, 2014; Aquilina et al., 2017). In terms of price informativeness, our results for the first trading period are consistent with the existing empirical result of Hendershott and Jones (2005), Comerton-Forde and Putniņš (2015) (for high levels of dark trading – i.e. higher than 10%), and Hatheway et al. (2017), while our results for the second trading period are related to the empirical results of Ready (2014) and Comerton-Forde and Putniņš (2015) (for low levels of dark trading). In addition, our model also provides new empirical implications regarding changes in market quality both in the time-series and the cross-section, emphasizing thus the role of stock characteristics for the decision of traders on whether to supply or demand liquidity in the exchange.

The paper is organized as follows. Section 2 presents the model. Section 3 analyzes the equilibria in a benchmark model without the dark pool. Section 4 presents the equilibria in the full model where rational traders have access to both the exchange and the dark pool. Section 5 analyzes how market quality indicators and investors' welfare change with the addition of a dark pool alongside an exchange and Section 6 provides the empirical implications of these results. Section 7 concludes. Proofs are presented in the Appendices.

2 Model

We consider a market in which a single risky asset is traded. The liquidation value of the asset, $\widetilde{V}$, may take two values, $V \in \{V^H, V^L\}$, with equal probabilities, where the unconditional mean of $\widetilde{V}$ is denoted by μ, and $\sigma > 0$ represents its volatility (i.e., its standard deviation). The asset may be traded in two venues: an exchange or a dark pool.

The exchange is organized as a limit order book (hereafter LOB). We assume that the initial LOB has three prices on the ask and bid sides of the book: A_1^1, A_1^2, A_1^3, and B_1^1, B_1^2, B_1^3, respectively, such that $V^L \leq B_1^3 < B_1^2 < B_1^1 < A_1^1 < A_1^2 < A_1^3 \leq V^H$. Prices are placed on a grid and the following relationships hold:

$$
\begin{aligned}
A_1^1 &= \mu + k_1\tau,\ A_1^2 = \mu + k_2\tau,\ A_1^3 = \mu + k_3\tau,\ V^H = \mu + \kappa\tau, \\
B_1^1 &= \mu - k_1\tau,\ B_1^2 = \mu - k_2\tau,\ B_1^3 = \mu - k_3\tau,\ V^L = \mu - \kappa\tau,
\end{aligned}
$$

with $1 \leq k_1 < k_2 < k_3 \leq \kappa$, where k_1, k_2 and k_3 are natural numbers, and τ is the tick size (i.e., the minimum price increment that traders are allowed to quote over the existing price). Note that the volatility of the asset satisfies $\sigma = \kappa\tau$, with κ a real number. For simplicity, we assume that the depth of the LOB at each bid and ask price is equal to 1, and that the LOB follows price and time priority rules.[8] The LOB is fully transparent (i.e., all the information in the LOB is available to all market participants at any point in time). Traders can submit market orders or limit orders to the LOB. There are no transaction costs.

[8] First, the order with the best price is executed. Second, among the orders with the same price, priority is given to the earliest arrival order.

The dark pool is completely opaque in the sense that an order submitted to the dark pool is not observable to anyone, but the trader who submitted it. We assume that the dark pool executes orders continuously. If an order is submitted to the dark pool and it is executed at time t, then the execution price is equal to the midpoint of the exchange at time t: $\frac{A_t^1 + B_t^1}{2}$. If the order is not executed in the dark pool at time t, then the trader can cancel it or keep it. If it is kept, then it returns to the exchange at $t+2$.[9]

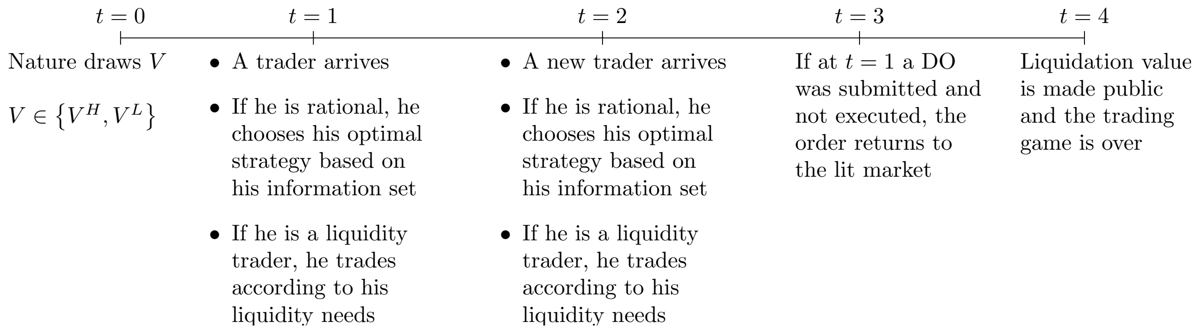

Figure 1: Timeline of the trading game when traders have access to the dark pool

The sequence of events illustrated in Figure (1) is as follows:

Date t=0: The liquidation value of the asset $\widetilde{V}$ is realized.

Dates t=1, 2: In each date, a *new* investor arrives to the market and observes the state of LOB. There are two possible types of traders: rational and liquidity traders. A rational trader decides whether or not to submit an order, the venue, and the type of order. By contrast, a liquidity trader always negotiates in the exchange and submits market orders in order to ensure immediate execution.

Date t=3: If a dark order submitted at $t = 1$ was neither executed nor canceled, then the order returns to the exchange.

Date t=4: The liquidation value of the asset is made public and the trading game is over.

For the sake of simplicity, we assume that all traders are risk neutral and may trade one unit of the asset. Rational traders choose an order submission strategy that maximizes their expected profits conditional on their information set at each date, I_t, which includes information about the liquidation value of the asset and about the state of the LOB. Rational traders simultaneously select whether or not to trade (NT), or if they trade, they choose the trading venue (exchange or dark pool), and the order type in the exchange (market order, MO, or limit order, LO). DO represents the order type in the dark pool. Consequently, the possible strategies of a rational trader (both

[9] In practice, the implementation of the decision to cancel or keep an order when it is not executed in the dark can be done using Smart Order Routing. Orders are automatically filled while sweeping for liquidity at the available venues.

informed and uninformed) are

$$\mathbb{O}_D = \{MO,\ LO,\ DO,\ NT\}, \tag{1}$$

where B in front of an order type denotes a buy order and S denotes a sell order.[10] The profits of a particular order are denoted by $\Pi^R_{\mathcal{O},t}$, where superscript R denotes that the order comes from a rational trader ($R = I, U$, where I is for the informed and U for the uninformed); subscript $\mathcal{O}$ is the order type $\mathcal{O} \in \mathbb{O}_D$ defined in (1); and subscript t is the date when the order is submitted.

Figure 2 illustrates the tree of events related to the first trading period.[11] A rational trader arrives to the market with probability $\lambda > 0$ and a liquidity trader arrives with probability $1-\lambda > 0$. Rational traders may be either (privately) informed if they have perfect information about the liquidation value of the asset (with probability $\pi > 0$), or (privately) uninformed if they only know the distribution of the liquidation value of the asset (with probability $1 - \pi$). We use $PIN \equiv \lambda\pi$ as a measure of information asymmetry following Easley and O'Hara (1987), and Easley et al. (1996). An informed trader buys whenever he observes $V = V^H$ (henceforth IH), and sells whenever he observes $V = V^L$ (henceforth IL). An uninformed trader is a buyer with probability $\frac{1}{2}$ (henceforth UB) and a seller with probability $\frac{1}{2}$ (henceforth US). Liquidity traders buy with probability $\frac{1}{2}$ or sell with probability $\frac{1}{2}$ for liquidity or hedging needs. The structure of the model and distributions of random variables are common knowledge. The final nodes of the tree include the profits for each of the actions profiles at $t = 1$.

For each possible order type, we next examine its characteristics and the associated expected profits for a rational buyer. Sell order profits are analogously defined. Internet Appendix I describes the expected profits of all traders at all times and for all the possible states of the LOB.[12]

- Market order (MO): It is executed immediately at the given best available ask/bid prices. The expected profits of a buy market order at date t are

$$\mathbb{E}\left(\Pi^R_{BMO,t}|I_t\right) = \mathbb{E}\left(\widetilde{V}|I_t\right) - A^1_t.$$

- Limit order (LO): A limit order that improves the current market price may be executed in the next period if a market order of the opposite sign hits the LOB. Thus, limit orders provide better prices than market orders but exhibit execution risk. We assume that a LO always improves the price by one tick because: (i) it is never optimal for the trader to improve the price by more than one tick since it reduces his profits; (ii) it is never optimal for the trader to submit a non-improving LO since the order is not executed (because the order goes to the queue and due to the time priority), and obtains zero profits. Given that we assume that there is also a discount factor, $\delta \in [0, 1]$, that is common across traders and periods, the expected

[10] For instance, BMO denotes a buy market order, while SMO a sell market order.

[11] A similar tree of events for the second trading period could be drawn.

[12] The Internet Appendices are available from the authors upon request.

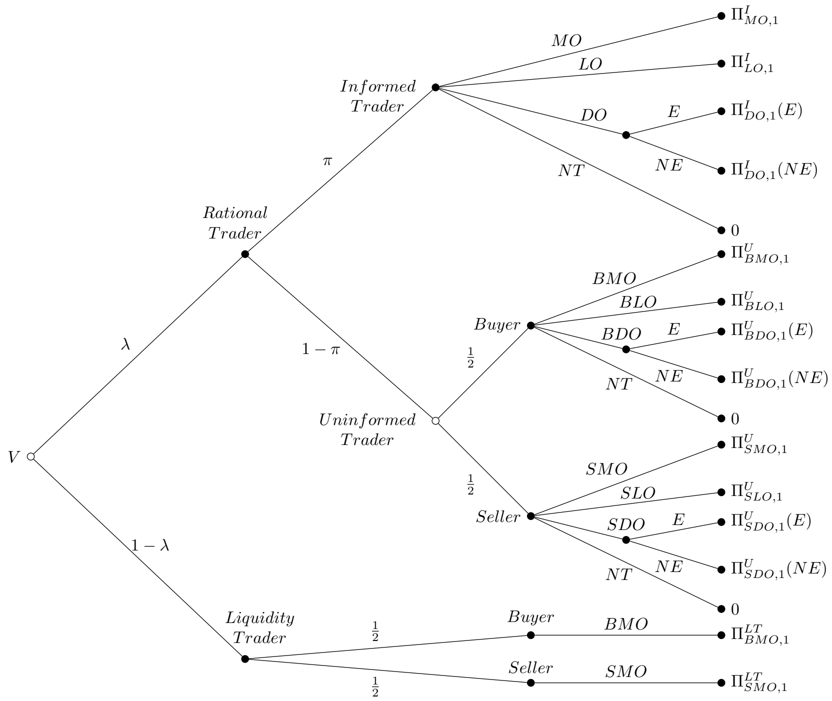

Figure 2: Tree of events of the first trading period.

profits of a buy limit order at date t are

$$\mathbb{E}\left(\Pi^R_{BLO,t}|I_t\right) = \delta p^R_{BLO,t}\left(I_t\right)\left(\mathbb{E}\left(\widetilde{V}|I_t\right) - B^1_t - \tau\right),$$

where $p^R_{BLO,t}$ is the probability of execution of a buy LO submitted by a rational trader at time t, respectively.

- Dark order (DO): With probability θ^R_t an order submitted by a rational investor to the dark pool at time t is executed, and with probability $(1-\theta^R_t)$ it is not executed. Since no new trader arrives in the market at $t = 3, 4$, an order that returns to the exchange from dark pool at $t+2$ will be either a MO (we name this dark order $BDO - MO$) or NT (we name this order $BDO - NT$). As the probability of execution of a LO at $t = 3$ is equal to 0, an order will never return to the market as a LO. We denominate the dark order DO as the best of the two dark orders $BDO - MO$ and $BDO - NT$ (for each type of trader).[13] Note that

[13] As it will be seen below, when an informed trader chooses at $t = 1$ to go to the dark pool, it is optimal for him that when the order is not executed to choose market order when returning to the exchange ($DO - MO$). When an uninformed trader chooses to go to the dark at $t = 1$, it is optimal for him that in case of non-execution the order to be canceled ($DO - NT$). At $t = 2$, as at $t = 4$ the game finishes, traders are indifferent between $DO - MO$ and $DO - NT$.

a DO does not change the state of the LOB, and to model the reporting delay of dark pool trades, we consider in our model that the dark pool does not report trades until the end of the trading game. Therefore, the expected profits of a buy dark order submitted at time t are

$$\begin{aligned}\mathbb{E}\left(\Pi^R_{BDO,t}|I_t\right) &= \max\left\{\mathbb{E}\left(\Pi^R_{BDO-MO,t}|I_t\right), \mathbb{E}\left(\Pi^R_{BDO-NT,t}|I_t\right)\right\}\\ &= \theta^R_t\left(\mathbb{E}\left(\widetilde{V}|I_t\right) - \frac{A^1_t + B^1_t}{2}\right) + (1-\theta^R_t)\delta^2\max\left\{\mathbb{E}\left(\Pi^R_{BMO,t+2}|I_t\right), 0\right\}.\end{aligned}$$

- No trade (NT): A trader who refrains from trading at t obtains zero profits :

$$\mathbb{E}\left(\Pi^R_{NT,t}|I_t\right) = 0.$$

In case of equality of profits, we assume that a MO dominates LO and DO; and a LO dominates DO. If the expected profits of a MO are null, a rational trader refrains from trading.

Our model can be represented by a sequential game of incomplete information and, therefore, we use the Perfect Bayesian Equilibrium equilibrium concept, hereafter PBE. In what follows, we focus on symmetric PBE in pure strategies. A symmetric equilibrium refers to a situation where buyers and sellers with the same information (i.e, informed or uninformed) choose the same type of order (except from the direction of trade).

3 Equilibrium in the Benchmark Model Without Dark Pool

We first consider the benchmark model without a dark pool (ND) where the available orders are: $\mathbb{O}_{ND} = \{MO, LO, NT\}$. The sequence of events can be seen in Figure (3). Note that the difference with respect to the timeline in Figure (1) when the dark pool is not available is that the liquidation value of the asset is revealed at $t = 3$.

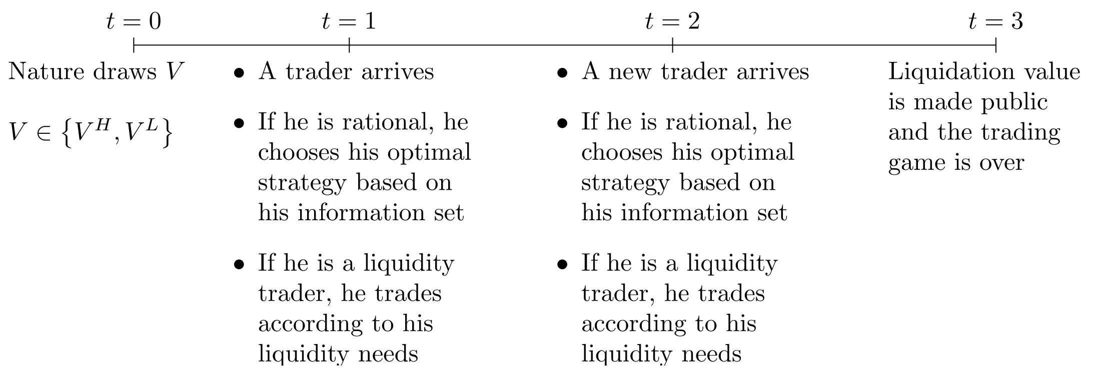

Figure 3: Timeline of the trading game when traders do not have access to the dark pool

We solve the game backwards. The expected profits for an informed and uninformed trader at $t = 2$ are summarized in Table A.1 and Table A.2, respectively, while Table A.4 and Table A.5

display the expected profits for rational traders at $t = 1$ (in Appendix A). The following lemma presents the informed and uninformed traders' optimal strategies at $t = 2$ and $t = 1$.

Lemma 1 *In equilibrium the following results hold:*

- *at $t = 2$: an informed trader always submits a MO, while an uninformed trader may submit either a MO or NT, but never chooses a LO.*
- *at $t = 1$: an informed trader may submit either a MO or a LO, but never chooses NT, while an uniformed trader may submit either a LO or NT, but never chooses a MO.*

An informed trader at $t = 2$ always chooses MO since it generates positive expected profits, while the expected profits of LO or NT are always null. In this trading period, an uninformed trader never chooses a LO since the probability of execution of this type of order is null given that no new orders arrive at $t = 3$. Consequently, the uninformed trader's choice at $t = 2$ will be either a MO or NT, which depends on the information gathered from the state of the LOB. When the state of the LOB conveys no information, then the optimal choice is NT since the expected profits of a MO are negative, as in the previous trading period. If the LOB reveals that at $t = 1$ a BMO or a BLO has been submitted, then the uninformed buyer at $t = 2$ chooses a BMO whenever his belief that the order at $t = 1$ came from an informed trader is sufficiently strong, which results in positive expected profits of submitting a BMO. Otherwise, the uninformed trader refrains from trading (NT). In addition, if the state of the LOB reveals that the trader at $t = 1$ submitted a SMO or a SLO, respectively, then the uninformed trader's expected profits of submitting a BMO at $t = 2$ are negative and, hence, the trader refrains from trading (NT).

At $t = 1$, an informed trader never chooses NT since it is always dominated at least by a MO and, hence, an informed trader may either choose a MO or a LO. In contrast, an uninformed trader at $t = 1$ never selects a MO since his expected profits are negative and, hence, it is always dominated at least by NT. Consequently, an uninformed trader at $t = 1$ may either choose LO or NT.

Hence, the candidate strategy profiles at $t = 1$ that can be sustained as a symmetric PBE are:

$$\mathcal{E}_1^{ND} : (BMO,\ SMO,\ BLO,\ SLO), \qquad \mathcal{E}_2^{ND} : (BMO,\ SMO,\ NT,\ NT),$$

$$\mathcal{E}_3^{ND} : (BLO,\ SLO,\ BLO,\ BLO), \qquad \mathcal{E}_4^{ND} : (BLO,\ SLO,\ NT,\ NT),$$

where the two first components correspond to strategies of informed traders at $t = 1$ (IH and IL, respectively) and the two last components correspond to strategies of uninformed traders at $t = 1$ (UB and US, respectively).

The next proposition characterizes the PBE of the reduced trading game where the dark pool is not available.

Proposition 1 *When there is no access to dark pool and $k_1 > 1$, the PBE of the game has the following optimal strategy profiles at $t = 1$:*

$$\begin{cases} (BLO, SLO, BLO, SLO) & \text{if } \ \sigma < \kappa^I_{MO-LO}\tau \text{ and } PIN < \psi^U_{LO-NT} \\ (BLO, SLO, NT, NT) & \text{if } \ \sigma < \kappa^I_{MO-LO}\tau \text{ and } PIN \geq \psi^U_{LO-NT} \\ (BMO, SMO, BLO, BLO) & \text{if } \ \kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN < \psi^U_{LO-NT}, \\ (BMO, SMO, NT, NT) & \text{if } \ \kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN \geq \psi^U_{LO-NT}, \end{cases}$$

where $\kappa^I_{MO-LO} \equiv (k_1 - 1) + 2\frac{\delta(k_1-1)(1-\lambda)+1}{2-\delta(1-\lambda)}$, $PIN \equiv \lambda\pi$ and $\psi^U_{LO-NT} \equiv \frac{(1-\lambda)(k_1-1)\tau}{\sigma-(k_1-1)\tau}$.

The optimal strategy of an informed trader at $t = 2$ is to choose MO for all possible states of the LOB. Table A.3 in Appendix A describes the optimal strategy of an uninformed trader at $t = 2$. The equilibrium beliefs of an uninformed trader at $t = 2$ are given in the proof of this proposition.[14]

Remark 1 *Notice that κ^I_{MO-LO} denotes the minimum value of κ such that at $t = 1$ an informed trader chooses MO instead of LO, PIN represents the probability of informed trading, while ψ^U_{LO-NT} represents the minimum value of PIN such that at $t = 1$ an uninformed trader chooses NT instead of LO.*

Proposition 1 shows that in the second period of trading, as explained also in Lemma 1, an informed trader submits a MO for all states of the LOB. An uninformed trader chooses NT, except if the state of the LOB conveys information about the fundamental value of the asset and he strongly believes that a MO or LO of the same direction as his order had been submitted at $t = 1$ by an informed trader. In such a case, this uninformed trader submits a MO.

In the first trading period, Proposition 1 indicates that when the asset's volatility is sufficiently low (i.e., $\sigma < \kappa^I_{MO-LO}\tau$), it is optimal for an informed trader to supply liquidity (i.e., to place a LO), while the decision of the uniformed trader depends on the severity of the adverse selection problem. Therefore, there are two possible optimal strategy profiles whenever the asset's volatility is low: (BLO, SLO, BLO, SLO) and (BLO, SLO, NT, NT). The optimal strategy profile (BLO, SLO, BLO, SLO) occurs in a market with low adverse selection risk (either because the asset's volatility is very low or the probability of informed trading is low), while (BLO, SLO, NT, NT) occurs whenever the adverse selection problem is sufficiently high (because the asset's volatility is not low or the probability of informed trading is high enough).[15] Combining the volatility and

[14] Proposition *7* in Appendix A characterizes the PBE when $k_1 = 1$. In this case, when the market is very liquid the only prevailing equilibrium strategy profile at $t = 1$ is (BMO, SMO, NT, NT) because neither an informed nor an uninformed trader finds it optimal to submit a LO.

[15] In particular, when the probability of informed trading is low, uninformed traders realize that, by setting LO at the exchange in the first trading period, they are very likely to end up trading with liquidity traders instead of informed traders, which leads them to trade.

information asymmetry dimensions, we call these stocks *"Low-Low"* (i.e., low volatility - low PIN), and *"Low-High"*, respectively.[16] In the subsequent analysis, we sometimes only consider one of these dimensions in isolation, such as low/high volatility stocks or low/high PIN.

By contrast, when the asset's volatility is sufficiently high (i.e., $\sigma \geq \kappa^I_{MO-LO}\tau$) it is optimal for the informed trader in the first trading period to demand liquidity (i.e., to place a MO). Note that the informational advantage of an informed trader increases with the volatility of the asset (σ). As a result, when the asset's volatility is sufficiently high, an informed trader prefers immediate execution (a MO). By contrast, when σ is not high enough, then the informed trader prefers a LO because of its price improvement. Furthermore, the decision of the uniformed trader depends again on the level of information asymmetry. Consequently, in this case, there are two possible optimal strategies: (BMO, SMO, BLO, BLO) and (BMO, SMO, NT, NT). The strategy (BMO, SMO, BLO, BLO) is optimal whenever the degree of information asymmetry is sufficiently low (i.e., $PIN < \psi^U_{LO-NT}$), while (BMO, SMO, NT, NT) is optimal whenever the degree of information asymmetry is sufficiently high (i.e., $PIN \geq \psi^U_{LO-NT}$). We call these stocks *"High-Low"*, and *"High-High"*, respectively.

Figure 4 illustrates these four possible regions and the optimal strategies at $t = 1$ that correspond to these equilibria. Note that the higher the asset's volatility, the lower the probability of informed trading needs to be for an uninformed trader to choose NT. These results derived in Proposition 1 are consistent with the previous work by Goettler et al. (2009) who show that informed traders switch from supplying to demanding liquidity when volatility changes from low to high.[17]

Interestingly, our model encompasses both the model of Zhu (2014) and Buti et al. (2017). Notice that when the volatility and PIN are high, i.e. a *"High-High"* stock, the optimal strategy for an informed trader at $t = 1$ is to place a MO as in Zhu (2014). Similarly, when the probability of having an informed trader is very small ($\pi \to 0$) the model is similar to Buti et al. (2017) where there is no asymmetric information. Notice that when $\pi \to 0$, an uninformed trader prefers a LO to NT, so we are in an equilibrium similar to $\mathcal{E}^{ND}_3$.

Remark 2 *Notice that a "High-Low" stock corresponds to equilibrium $\mathcal{E}^{ND}_1$; a "High-High" to $\mathcal{E}^{ND}_2$; a "Low-Low" to $\mathcal{E}^{ND}_3$; and a "Low-High" to $\mathcal{E}^{ND}_4$.*

The following corollary describes the comparative statics of κ^I_{MO-LO} and ψ^U_{LO-NT} with respect to various parameters.

Corollary 1 *Ceteris paribus, κ^I_{MO-LO} increases with δ and k_1, and decreases with λ, while ψ^U_{LO-NT} increases with k_1, and decreases with λ and κ.*

The previous corollary implies that for the informed trader at $t = 1$, an increase in the discount factor, a decrease in the liquidity of the asset ($1/k_1$), or an increase in the probability that a liquidity

[16] Simulations show that the region *"Low-High"* (and, therefore, the strategy profile (BLO, SLO, NT, NT)) is obtained for very specific parameter configurations. In particular, we find that this can only occur when $\pi > 0.5$.

[17] Goettler et al. (2009) point out that, first, as the volatility increases, the risk of a limit order increases as they are more likely to picked-off for trading. Second, as the volatility increases, it is more likely to find mispriced orders in the limit order book.

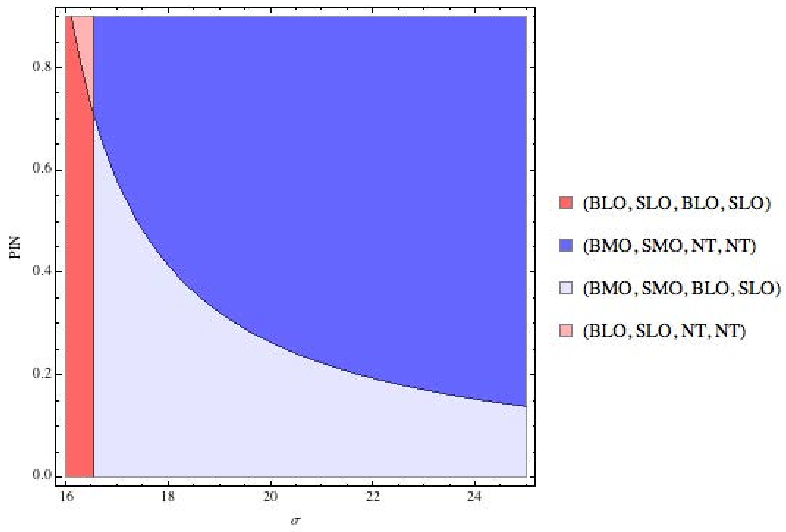

Figure 4: Optimal strategies at $t = 1$ without dark pool. Parameters values: $k_1 = 30$, $k_2 = 31$, $\lambda = 0.9$, $\tau = 0.5$, $\delta = 1$.

trader arrives at $t = 2$ (ceteris paribus) reduces the relative attractiveness of a MO with regards to a LO.[18] With regards to the uninformed trader at $t = 1$, a decrease in the liquidity of the asset, an increase in the probability that a liquidity trader arrives at $t = 2$, or a reduction in the volatility of the asset at $t = 2$ (ceteris paribus) increases the attractiveness of a LO with regards to a NT.

Note that according to Corollary 1 the condition $\sigma < \kappa^{I}_{MO-LO}\tau$ can be satisfied, ceteris paribus, in a low volatility stock, or in a low liquidity stock (high k_1), or when rational traders are characterized by low immediacy (high δ) or when they participate in a small proportion to the market (small λ). In addition, notice that our classification of high/low volatility stocks also depends on the tick size: ceteris paribus, as the tick size increases the low volatility region expands. To sum up, the characterization of stocks as *"High"* and "*Low*" in terms of liquidity, immediacy or proportion of rational traders gives analogous results to the characterization in terms of *"High"* and "*Low*" volatility. For the sake of simplicity, we exemplify our analysis by discussing it in terms of the asset's volatility, but a similar analysis may be done by studying changes in other stock market and trader characteristics.

4 Equilibrium in a Model With the Dark Pool

We next consider a model where rational traders have both access to the exchange and to the dark pool. Hence, the orders that can be submitted are given in (1).

The decision to submit an order to the dark pool depends on its probability of execution. Let $\widetilde{z}$

[18]The profits of an informed trader do not depend on the probability that an informed trader arrives in the next trading period, $\lambda\pi$. This is due to the fact that an informed trader that submits a LO at $t = 1$ knows that the LO will not be executed in the next trading period against an order submitted by an informed trader since an informed trader at $t = 2$ chooses an order of the same sign as the initial order. In addition, an informed trader at $t = 1$ correctly predicts that an uninformed trader at $t = 2$ never submits a MO of the opposite sign as the informed trader at $t = 1$.

be a random variable representing the order imbalance of the dark pool at the beginning of $t = 1$. Suppose that $\widetilde{z}$ is symmetric such that the probability of execution of a dark order of size 1 for an rational trader at $t = 1$ is $\theta_1^{R,1} = pr_R(\widetilde{z} \geq 1) = pr_R(\widetilde{z} \leq -1)$, and correspondingly for an order of size 2 is $\theta_1^{R,2} = pr_R(\widetilde{z} \geq 2) = pr_R(\widetilde{z} \leq -2)$, where rational traders may be informed or uninformed, $R = I, U$. To simplify notation, let θ_t^I and θ_t^U be the probability of execution of a *DO* at trading period t for an informed and uninformed trader, respectively. In particular, $\theta_1^R \equiv \theta_1^{R,1}$, for $R = I, U$. In addition, note that the probability of execution in the dark pool is endogenous in the second trading period since it depends on the actions of traders at $t = 1$.

The following lemma describes the relationship between the probability of execution in the dark at $t = 2$ for informed and uninformed traders.

Lemma 2 *Suppose that the prior probability distribution of the order imbalance in the dark pool is common for all rational traders. Then, the probability of execution of a dark pool order in the second trading period is greater or equal for an uninformed than for an informed trader, i.e.* $\theta_2^I \leq \theta_2^U$.

The reasoning for Lemma 2 is as follows. Suppose that informed and uninformed traders face the same probability of execution in the dark pool at $t = 1$ (i.e., $\theta_1^I = \theta_1^U$). If in the first trading period a trader chooses a dark order, then the informed investor at $t = 2$ is aware that the direction of his order and the direction of the order submitted by an informed trader at $t = 1$ are the same. However, the coincidence of order direction between periods does not have to be fulfilled for an uninformed investor. Hence, since the probability of execution of a dark order of size 1 is greater than the corresponding one of an order of size 2, it is less likely that a dark order of an informed trader is executed in the dark at $t = 2$ compared to that of an uninformed trader ($\theta_2^I < \theta_2^U$). This difference ($\theta_2^U - \theta_2^I$) increases with the probability that an informed trader arrives at the market at $t = 1$ ($\lambda\pi$). We also find that when no trader chooses a dark pool order at $t = 1$, then the probability of execution in the dark pool at $t = 2$ is the same for informed and uninformed traders ($\theta_2^I = \theta_2^U$) provided that $\theta_1^I = \theta_1^U$. This result is similar to the mechanism in Zhu (2014) although in our setup the informed trader may go the dark pool.

As in the previous section, we solve the model backwards. First, we calculate the expected profits for an informed and an uniformed trader at $t = 2$ and $t = 1$ as summarized in Tables B.1, B.2, B.3 and B.4, respectively in Appendix B. Comparing the expected profits of each of the possible orders for each type of rational trader at $t = 2$ and $t = 1$, Lemma 3 states the strategies that are dominated and, hence, never chosen by a rational player.

Lemma 3 *In equilibrium the following results hold:*

- *at $t = 2$: an informed trader may submit either a MO or a DO, but never chooses a LO or NT, while an uninformed trader may submit either a MO, a DO or NT, but never chooses a LO.*

- *at $t = 1$: an informed trader may submit either a MO, a LO or a DO, but never chooses NT, while an uninformed trader may submit either a LO or NT, but never chooses a MO or a DO.*

We find that in the second trading period a LO is never chosen since it is never executed: a) if the LOB has changed, then no MO arrives at $t = 3$ and, hence, it has zero probability of execution; b) if the LOB has not changed, then LO can only be executed if an uninformed trader at $t = 1$ chooses a DO, but as we explain in the next paragraph this cannot occur in equilibrium since its expected profits are negative.[19] In addition, an informed trader at $t = 2$ never chooses NT since it is always dominated by a MO.

In the first trading period, an informed trader never chooses NT since it is always dominated by at least a MO. By contrast, an uninformed trader at $t = 1$ may choose between a LO or NT since the expected profits of a MO and a DO are negative. Let us discuss why the expected profits of a DO submitted at $t = 1$ by an uninformed trader are negative (see Table B.4), and hence this order is never chosen. If a DO is executed at $t = 1$, then its expected profits are zero, which occurs with probability θ_1^U. If the order is not executed at $t = 1$ and returns to the market at $t = 3$, which occurs with probability $1 - \theta_1^U$, expected profits depend on whether the uninformed trader who returns to the exchange decides to submit a NT or MO. If the uninformed trader selects NT then expected profits are also equal to zero, while if he submits a MO, then the expected profits are always negative. Note that, for an uninformed buyer, the difference between a BMO at t=1 and a BMO at t=3 are the potential differences in prices. The trader might see an improvement in the price due to the possibility that at $t = 2$ a rational trader arrives and submits a SLO, but he might see a deterioration in the price due to the possibility that a trader (rational or a liquidity) at $t = 2$ submits a BMO. We find that the increase in expected profits due to the price improvement is not large enough and, consequently, the expected profits of a DO returning to the exchange at $t = 3$ as a MO are strictly negative as are the ones corresponding to a MO at $t = 1$. In contrast, the expected profits of a DO submitted by an informed trader at $t = 1$ are strictly positive (see Table B.3) and, hence, a DO might be optimal for the informed trader at $t = 1$.

Hence, the candidate strategy profiles at $t = 1$ that can be sustained as a PBE are:

$$\begin{aligned} &\mathcal{E}_1^D : (BMO, SMO, BLO, SLO), \quad && \mathcal{E}_2^D : (BMO, SMO, NT, NT), \\ &\mathcal{E}_3^D : (BLO, SLO, BLO, BLO), && \mathcal{E}_4^D : (BLO, SLO, NT, NT), \\ &\mathcal{E}_5^D : (BDO, SDO, BLO, SLO), && \mathcal{E}_6^D : (BDO, SDO, NT, NT), \end{aligned}$$

where, as before, the two first components correspond to strategies of informed traders at $t = 1$ (IH and IL, respectively) and the two last components correspond to strategies of uninformed traders at $t = 1$ (UB and US, respectively).

[19]The uninformed trader at $t = 2$ forms the correct beliefs that, if a LO is executed at $t = 3$, it must have come from an informed trader at $t = 1$ with probability 1. But this information reveals to the uninformed buyer (seller) that the value of the asset must be low (high) and, hence, expected profits of a LO are negative.

The equilibrium of the trading game where rational traders have access to a dark pool is characterized in the following proposition.

Proposition 2 *If $k_1 > 1$, then a PBE of the game has the following optimal strategies:*

Case 1 *If $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BLO, SLO, BLO, SLO) & \text{if } \theta^I_1 \text{ is sufficiently small,} \\ (BDO, SDO, BLO, SLO) & \text{if } \theta^I_1 \text{ is sufficiently large.} \end{cases}$$

Case 2 *If $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN \geq \psi^U_{LO-NT}$, then the optimal strategy profile at $t = 1$ are:*

$$\begin{cases} (BLO, SLO, NT, NT) & \text{if } \theta^I_1 \text{ is sufficiently small,} \\ (BDO, SDO, NT, NT) & \text{if } \theta^I_1 \text{ is intermediate,} \\ (BDO, SDO, BLO, SLO) & \text{if } \theta^I_1 \text{ is sufficiently large.} \end{cases}$$

Case 3 *If $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN < \psi^U_{LO-NT}$, then the optimal strategy profiles at $t = 1$ are:*

$$\begin{cases} (BMO, SMO, BLO, SLO) & \text{if } \theta^I_1 \text{ is sufficiently small,} \\ (BDO, SDO, BLO, SLO) & \text{if } \theta^I_1 \text{ is sufficiently large.} \end{cases}$$

Case 4 *If $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^U_{LO-NT}$, then the optimal strategy profile at $t = 1$ is*

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if } \theta^I_1 \text{ is sufficiently small,} \\ (BDO, SDO, NT, NT) & \text{if } \theta^I_1 \text{ is intermediate,} \\ (BDO, SDO, BLO, SLO) & \text{if } \theta^I_1 \text{ is sufficiently large.} \end{cases}$$

The proof in Appendix B characterizes the threshold values of θ^I_1 for which each strategy profile is optimal at $t = 1$.

The informed and uninformed traders' optimal choices at $t = 2$, and beliefs of the uninformed traders at $t = 2$, are characterized in Lemma 4 in Appendix B.[20]

We start backwards by discussing the second trading period. An informed trader submits MO (DO) for all states of the LOB when the execution risk in the dark is sufficiently high (low) in relation to the price improvement offered by a DO. As the execution risk in the dark lowers, an informed buyer replaces a BMO by a BDO in the following order of states of the LOB: $(A^2_1, B^1_1), (A^1_1, B^2_1), (A^1_1, B^1_1), (A^1_1, B^1_1 + \tau), (A^1_1 - \tau, B^1_1)$. This is because, when a BMO had been submitted at $t = 1$, the gain from another BMO is small in relation to a BDO even though the execution risk in the dark is relatively high. However, when a SLO had been previously submitted at $t = 1$, then the gain from a BMO is large in relation to a BDO despite the execution risk in the dark is relatively low.

[20]Proposition 8 in Appendix B characterizes the PBE when $k_1 = 1$.

For an uninformed trader at $t = 2$, the optimal strategy critically depends on his set of beliefs about the probability that a MO, LO or DO order was submitted by an informed trader at $t = 1$. When the state of the LOB contains no information, i.e., (A_1^1, B_1^1), then an uninformed trader at $t = 2$ chooses NT since the expected profits of a MO are negative, and the expected profits of a DO are zero because the mid-point price is equal to the unconditional expected value of the asset. However, an uninformed trader may also choose a MO or DO in the second trading period if the book is sufficiently informative about the fundamental value of the asset. Notice that, in contrast with the first trading period, if the probability of execution in the dark at $t = 2$ is sufficiently high, then an uninformed trader may migrate to the dark venue.

In the first trading period, Proposition 2 shows that having access to a dark pool changes the optimal submission strategy profiles for informed and uninformed traders. When the probability of execution in the dark for informed traders at $t = 1$ (i.e., θ_1^I) is sufficiently high, then an informed trader switches the trading venue: from the exchange to the dark pool. Otherwise, if the execution risk in the dark pool is sufficiently high, then an informed trader submits the same type of orders (MO or LO) to the exchange as in the benchmark model without the dark pool. The threshold values of the probability of execution in the dark reflect the price improvement and execution trade-off of each type of order. In case of execution, the best price is achieved by a LO, followed by a DO, and the worst price is given in a MO. While a LO has execution risk, MO and DO do not face execution risk for an informed trader. Note that at $t = 1$ a DO faces no risk of execution since we find that it is optimal for an informed trader to determine that if the DO that is not executed in the first trading period then he routes it back to the exchange as a MO at t=3. However, when this order returns to the exchange at $t = 3$, it faces the risk that the price has worsened because of the order submitted by the trader that arrives at $t = 2$. Numerical simulations show that the cutoff θ_1^I that is needed for an informed trader to switch to the dark pool is generally larger when the volatility is larger.

We find that an uninformed trader never goes to the dark venue in the first trading period as discussed in Lemma 3. However, the introduction of the dark pool might change the optimal strategy of an uninformed trader when the probability of execution in the dark for an informed trader is high enough since an uninformed trader may switch from NT to LO. This is because the low execution risk in the dark pool encourages an informed trader at $t = 2$ to trade in the dark pool rather than in the exchange. Consequently, adverse selection is reduced in the exchange at $t = 1$, which promotes trading of an uninformed trader.

Proposition 2 suggests that restricting the informed trader to participate in the dark pool might harm the uninformed trader. To illustrate this point, notice that Cases 2 and 4 of this proposition show that a significant reduction of θ_1^I might discourage the uninformed trader to participate in the exchange in the first trading period.

Figure 5 illustrates the optimal strategies at $t = 1$ with respect to the asset's volatility and the probability of execution for the informed trader in the dark (θ_1^I) for selected parameter values. When θ_1^I is small, then the optimal strategy profiles at $t = 1$ coincide with the ones in the benchmark

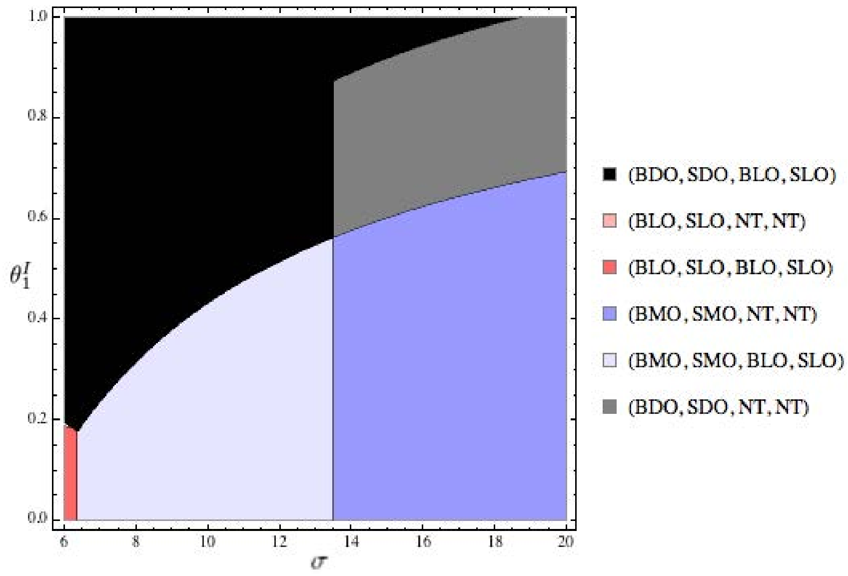

Figure 5: Optimal strategies at $t = 1$ with dark pool. Parameters values: $k_1 = 10$, $k_2 = 11$, $\lambda = 0.5$, $\pi = 0.5$, $\tau = 0.5$, $\delta = 0.5$, and $\theta_1^{I,2} = \theta_1^{U,2} = 0$. In addition, we have assumed beliefs such that at $t = 2$ an uninformed seller does not select a SLO when there is no change in the prices of the LOB, and that an informed buyer chooses a BMO at $t = 2$ when the LOB has not changed.

model without the dark pool. When an informed trader chooses a MO at $t = 1$, the graph shows that the threshold value of θ_1^I that generates a migration of an informed trader's MO to a DO increases with the asset's volatility. In contrast, when the informed trader chooses LO at $t = 1$, the threshold value of θ_1^I that generates a switch from the LO to a DO decreases with the asset's volatility, and it is small. We find that, generally, LO are the first to go to the dark pool since they are more similar to DO in the immediacy and price-improvement trade-off. Additionally, we observe that the probability θ_1^I that makes an uninformed trader switch from NT to LO is large and it is increasing with the volatility.

Figure 6 illustrates the optimal strategies at $t = 1$ with respect to the asset's volatility and information asymmetry for several values of $\theta_1^I \in \{0.1, 0.3, 0.5, 0.8\}$, which are displayed in panels a), b), c) and d), respectively. In Panel a), when $\theta_1^I = 0.1$, the graph has the same features as in Figure 4 since for small values of θ_1^I there is no migration to dark pool. The illustration of Panel b), when $\theta_1^I = 0.3$, shows that the first few informed traders' orders that migrate to the dark are those corresponding to low levels of the asset's volatility irrespective of the level of information asymmetry: (BLO, SLO, BLO, SLO) migrates to (BDO, SDO, BLO, SLO) and, (BMO, SMO, BLO, SLO) to (BDO, SDO, BLO, SLO). Panel c), with $\theta_1^I = 0.5$, shows the same trend as Panel b), and also that, for low levels of volatility and high levels of information asymmetry, the strategy (BMO, SMO, NT, NT) switches to (BDO, SDO, NT, NT). In Panel d), with $\theta_1^I = 0.8$, we find that the informed trader has completely migrated to the dark and the only two optimal strategy profiles are (BDO, SDO, BLO, SLO) and (BDO, SDO, NT, NT).

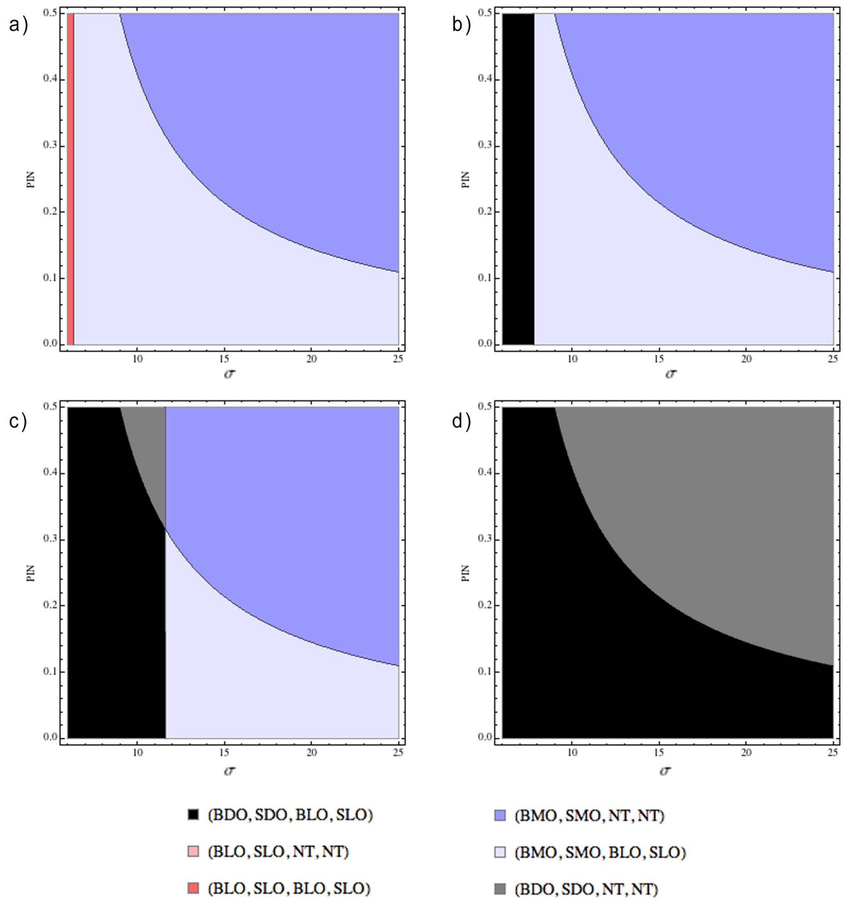

Figure 6: Optimal strategies at $t = 1$ with dark pool. Parameters values: $k_1 = 10$, $k_2 = 11$, $\lambda = 0.5$, $\pi = 0.15$, $\tau = 0.5$, $\delta = 0.5$, and $\theta_1^{I,2} = \theta_1^{U,2} = 0$. Panel a) has $\theta_1^I = 0.1$, panel b) has $\theta_1^I = 0.3$, panel c) has $\theta_1^I = 0.5$, and panel d) has $\theta_1^I = 0.8$. In addition, we have assumed beliefs such that at $t = 2$ an uninformed seller does not select a SLO when there is no change in the prices of the LOB, and that an informed buyer chooses a BMO at $t = 2$ when the LOB has not changed.

5 Market Quality and Welfare

In this section, we examine how the introduction of the dark pool affects market quality and welfare. To do so, we compare the following measures of market quality with and without a dark pool in both trading periods: price informativeness, expected inside spread, expected volume, and trade creation. Additionally, we present results for welfare, measured by the expected profits of market participants.

In this section, the signs of the market quality and welfare comparisons may depend on market characteristics and trading period, t. In order to show the signs of the comparisons of the different measures in a compact form, we use the following table format:

Market quality parameter	t					
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$	X				X	
$\mathcal{E}_2^{ND}$		X			X	X
$\mathcal{E}_3^{ND}$			X		X	
$\mathcal{E}_4^{ND}$				X	X	X

.

The rows of the table show the prevailing equilibria when the dark pool is unavailable (Proposition 1), while the columns of the table display the equilibria when the exchange and the dark pool coexist (Proposition 2). The filled cells in the table show the possible transitions from the prevailing equilibria without the dark pool to the equivalent ones when the dark pool is introduced, as shown in Proposition 2. The empty cells mean that the comparison is not meaningful (as the transition between these equilibria is not possible). The potential symbols in the comparisons are: "$=$", "$<$", "$\leq$", "$>$", "$\geq$". The sign "$=$" means that the market quality measure at t is identical with and without access to the dark pool; a "$<$" ("$\leq$") shows that the expected market quality parameter at t corresponding to $\mathcal{E}_i^{ND}$ is lower (lower or equal) than the market quality parameter at t corresponding to $\mathcal{E}_j^D$; and the reverse for "$>$" ("$\geq$"). The same applies for welfare comparisons.

We study first price informativeness. Price informativeness in a trading period can be measured by the expected difference between the unconditional variance of the value of the asset and the conditional variance of the value of the asset given the set of prices at the end of such a trading period. As Lemma 3 indicates, in the first trading period only an informed trader might migrate to the dark pool, while in the second trading period both informed and uninformed traders might trade in the dark venue. The change in dark pool's order attractiveness for uninformed traders between the first and the second trading periods brings about differences in how adding a dark pool affects price informativeness in both trading periods, as shown in the next proposition.

Proposition 3 (Price informativeness) *Adding a dark pool alongside an exchange:*

(i) in the first trading period, price informativeness is lower or remains the same;

(ii) in the second trading period, price informativeness decreases when there is transition from equilibrium $\mathcal{E}_1^{ND}$ to $\mathcal{E}_1^D$ and is ambiguous otherwise.

In the first trading period, price informativeness is lower with the introduction of the dark pool if the informed trader's execution probability in the dark pool at $t = 1$ is sufficiently large. In this case, since the informed trader migrates to the dark pool and, hence, the informational content of price is reduced. In the rest of the cases, price informativeness remains the same.

In the second trading period, price informativeness might be higher, lower or the same with a dark pool compared with the case without it. Let us illustrate these possibilities with an example. Consider the equilibrium where at $t = 1$ an informed trader chooses a MO and an uninformed trader decides NT both with and without access to the dark pool. If the informed and uninformed traders choose different trading venues at $t = 2$, we have contrasting results regarding price informativeness.

If the informed trader goes to the dark pool, while the uninformed remains in the exchange (placing a MO in some states of the LOB), then we expect a reduction in price informativeness, analogously to the first trading period. By contrast, when the informed stays in the exchange (placing a MO) and the uninformed migrates to the dark pool, then we expect an increase in price informativeness in the second trading period. It is noteworthy that the ambiguity in the comparison of price informativeness at $t = 2$ occurs in all equilibria, except in the case there is transition from equilibrium $\mathcal{E}_1^{ND}$ to $\mathcal{E}_1^{D}$. In $\mathcal{E}_1^{ND}$, at $t = 2$ uninformed traders choose NT and, hence, we never expect an increase in price informativeness with the introduction of a dark pool.

The next proposition shows how the access to a dark pool affects the ex-ante expected inside spread in the exchange, denoted by $\mathbb{E}_0(S_t)$, in each trading period.

Proposition 4 (Expected inside spread) *Due to the introduction of the dark pool, the transition from equilibrium $\mathcal{E}_i^{ND}$ to $\mathcal{E}_j^{D}$ (for $i = 1,..,4$ and $j = 1,...,6$, respectively) has the following effects on ex-ante expected spreads in both trading periods:*

$\mathbb{E}_0(S_t)$	$t=1$						$t=2$					
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$	$=$				$>$		$\geq$				$>$	
$\mathcal{E}_2^{ND}$		$=$			$>$	$>$		$\geq$			$>$	$>$
$\mathcal{E}_3^{ND}$			$=$		$<$				$\geq$		$\geq \leq$	
$\mathcal{E}_4^{ND}$				$=$	$<$	$<$				$\geq$	$>$	$\geq \leq$

In the first trading period, Propositions 2 and 4 show that the change in the expected inside spread depends on market characteristics. The results can be explained by noting that the switch from MO to DO reduces the inside spread, while the switch from LO to DO increases the inside spread, and the switch from NT to LO reduces the inside spread. Hence, if the introduction of the dark pool makes the informed trader switch from MO to DO, then the expected inside spread is reduced regardless of the behavior of the uninformed trader. However, when the informed trader switches from LO to DO, then the change in the expected inside spread is strictly larger with the introduction of the dark pool. If the uninformed trader does not change his trading strategy, then the expected inside spread is unambiguously larger with the dark pool. However, if the uninformed trader switches from NT to LO then its effect is potentially ambiguous. On the one hand the spread increases because the informed trader does not supply liquidity anymore to the LOB, but on the other hand the spread might decrease because now is the uninformed trader the one who supplies liquidity on the LOB. However, note that E_4^{ND} prevails when the degree of information asymmetry is sufficiently high, and in this case the effect of the informed trader on spread dominates the one of the uninformed. Consequently, the expected inside spread is always strictly higher with the dark pool than without it.

At the beginning of the second trading period we could have different spreads depending on whether the dark pool is available or not. If we have a higher or equal inside spread at the beginning of $t = 2$ without the dark pool, then having access to the dark pool unambiguously reduces the ex-ante expected inside spread. This is because, at $t = 2$ an informed trader might switch from MO to DO and an uninformed trader from MO or NT to DO, which reduce the expected inside spread. However, if we expect a lower inside spread at the beginning of $t = 2$ without the dark pool, then we obtain ambiguous results. The possibility of submitting a DO in the second trading period might reduce the inside spread, which goes in the opposite direction to the one obtained in the first trading period.

We next analyze how the introduction of the dark pool changes expected traded volume in the exchange, denoted by $\mathbb{E}_0(V_{EX,t})$, and expected traded volume in the dark pool in both trading period. Using expected total trading volume in period t, denoted by $E_0(V_{T,t})$, which aggregates the volume in the two venues, we can infer if there is trade creation or destruction depending on whether the total trading volume increases or decreases, respectively.

Proposition 5 (Expected volume and trade creation) *Due to the introduction of the dark pool:*

a) The transition from $\mathcal{E}_i^{ND}$ to $\mathcal{E}_j^D$ (for $i = 1,..,4$ and $j = 1,...,6$) has the following effects on the ex-ante expected volume in the exchange at $t = 1$:

$\mathbb{E}_0(V_{EX,1})$	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$	$=$				$>$	
$\mathcal{E}_2^{ND}$		$=$			$>$	$>$
$\mathcal{E}_3^{ND}$			$=$		$=$	
$\mathcal{E}_4^{ND}$				$=$	$=$	$=$

,

and at $t = 2$, the ex-ante expected volume in the exchange is always lower or equal when there is access to the dark pool.

At $t = 1$ and $t = 2$, the ex-ante expected volume in the dark pool is strictly larger if market conditions are such that order migration to the dark pool occurs in the given period.

b) At $t = 1$, the transition from $\mathcal{E}_i^{ND}$ to $\mathcal{E}_j^D$ (for $i = 1,..,4$ and $j = 1,...,6$) has the following effects on trade creation or destruction:

$\mathbb{E}_0(V_{T,1})$	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$	$=$				$>$	
$\mathcal{E}_2^{ND}$		$=$			$>$	$>$
$\mathcal{E}_3^{ND}$			$=$		$<$	
$\mathcal{E}_4^{ND}$				$=$	$<$	$<$

At $t = 2$, there might be trade creation or destruction in each of the possible equilibria comparisons.

In the first trading period, due to the introduction of the dark pool expected trading volume in the exchange remains the same as when the dark pool is not available, except if the informed trader switches from MO to DO. In this last case provided that θ_1^I is sufficiently large, ex-ante expected volume in the exchange decreases. This is because the informed trader's MO migrates to the dark, while the uninformed trader either does not change his type of order or switches from NT to LO, which does not cause any change in the exchange's trading volume. In terms of expected total trading volume, we find that it remains the same if there is no order migration towards the dark pool, it decreases if the informed trader switches from a MO to a DO, and it increases if the informed trader switches from a LO to a DO.[21] Total expected trading volume decreases when θ_1^I is sufficiently large so that the informed switches from a MO to a DO since orders that are submitted to the dark pool do not execute at $t = 1$ with probability $1 - \theta_1^I$. In contrast, when the introduction of the dark pool makes the informed trader switch from LO to DO there is trade creation in this trading period because the expected volume in the dark increases.

In the second period of trading, the addition of the dark pool makes the expected trading volume in the exchange smaller or equal compared to when the dark pool is unavailable. This is because at $t = 2$, when the dark pool is unavailable, the informed trader chooses MO, while the uninformed trader MO or NT. However, if the dark becomes available and market conditions are favorable, both traders may also choose to trade in the dark. Hence, expected trading volume in the exchange is either lower or remains unchanged due to the introduction of the dark pool.

At $t = 2$, we find that there might be trade creation or destruction in each of the possible equilibria comparisons. Let us discuss the rationale of this ambiguous result. An informed trader that does not have access to the dark pool always chooses a MO at $t = 2$, while with the dark pool there is a potential migration to the dark pool by the informed trader, which destroys trade in the second trading period. An uninformed trader without access to the dark pool always chooses a MO or NT, while with the dark pool there is a potential migration to the dark pool. If there is migration from a MO to a DO there is trade destruction, while if there is a switch from NT to DO there is trade creation at $t = 2$. Hence, if introducing the dark pool makes both informed and uninformed traders choose DO instead of MO in some states of the LOB, then there is trade destruction.[22] However, we could have trade creation if the uninformed switches from NT to DO and the probability that an uninformed trader arrives is large or if the informed trader stays in the exchange because θ_1^I is sufficiently small.

In the next proposition we consider how adding a dark pool affects market participants' welfare: the unconditional expected profits of informed traders, uninformed traders, and liquidity traders in both trading periods. Define $\mathbb{E}_0(\Pi_{t,n})$ with $n = U, LT$ as the ex-ante expected profits in the trading period t for an uninformed and a liquidity trader, respectively.

[21] Note that here we only consider trade creation or destruction at $t = 1$. However, when we consider trade creation or destruction across periods, we would find that there is the same amount of trade both in the case with or without dark pool, if the informed chooses a MO. This is because, if a informed's MO switches to a DO and it is not executed in the dark, then it returns as a MO at $t = 3$.

[22] Note that our game is truncated at $t = 3$, and only dark orders submitted at $t = 2$ and are executed are taken into account.

Proposition 6 (Welfare: Expected profits) *Due to the introduction of the dark pool:*

i) At $t = 1$, the unconditional expected profits of an informed trader remain the same if there is no order migration to the dark, and they are strictly larger if market conditions are such that the orders of the informed trader migrate to the dark. At $t = 2$, the profits of an informed trader are always larger or equal with access to the dark pool than without it.

ii) The unconditional expected profits of an uninformed trader in both trading periods. The transition from equilibrium strategy profiles $\mathcal{E}_i^{ND}$ to $\mathcal{E}_j^{D}$ (for $i = 1, .., 4$ and $j = 1, ..., 6$, respectively) has the following effects:

$\mathbb{E}_0(\Pi_{t,U})$	$t=1$						$t=2$					
	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$	$=$				$\leq$		$<$				$<$	
$\mathcal{E}_2^{ND}$		$=$			$<$	$=$		$\leq$			$\geq \leq$	$\geq \leq$
$\mathcal{E}_3^{ND}$			$=$		$\leq$				$<$		$\geq \leq$	
$\mathcal{E}_4^{ND}$				$=$	$<$	$=$				$<$	$\geq \leq$	$\geq \leq$

iii) Liquidity traders: At $t = 1$ the unconditional expected profits remain unchanged due to the introduction of the dark pool, while at $t = 2$ the transition from an equilibrium strategy profiles $\mathcal{E}_i^{ND}$ to $\mathcal{E}_j^{D}$ (for $i = 1, .., 4$ and $j = 1, ..., 6$, respectively) has the following effects:

$\mathbb{E}_0(\Pi_{2,LT})$	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_3^D$	$\mathcal{E}_4^D$	$\mathcal{E}_5^D$	$\mathcal{E}_6^D$
$\mathcal{E}_1^{ND}$	$=$				$<$	
$\mathcal{E}_2^{ND}$		$=$			$<$	$<$
$\mathcal{E}_3^{ND}$			$=$		$>$	
$\mathcal{E}_4^{ND}$				$=$	$>$	$>$

In the first trading period, expected profits of each type of market participant are not lower with the introduction of the dark pool compared to when the dark pool is unavailable. First, an informed trader strictly increases his profits since the price improvement obtained by submitting a DO outweighs the execution risk in the dark. Second, even if an uninformed trader does not go to the dark at $t = 1$, he has larger profits under certain market conditions. This is because the migration of the informed trader's orders to the dark pool reduces information asymmetry in the LOB. As a result, the probability of an uninformed trader facing an informed trader is smaller with the dark pool. Therefore, the profits of an uninformed trader are higher or equal than when the dark pool is not available. Moreover, when the uninformed switches from NT to a LO then his profits are strictly larger. Third, the profits of a liquidity trader remain the same since he does not change his trading strategy as he cannot trade in the dark pool.

In the second trading period, expected profits of an informed trader are not lower due to the introduction of the dark pool. However, the expected profits of an uninformed trader are ambiguous, and might even be lower with the dark pool, except in the case where market conditions are such that the prevailing equilibrium at $t = 1$ is $\mathcal{E}_1^{ND}$. In this last case, the expected profits of an uninformed trader are always larger with the dark pool. Let us discuss the rationale for an ambiguous case. By looking at the state of the LOB, the uninformed trader can extract information about the value of the asset, which might make him set a MO at $t = 2$ when the dark pool is unavailable. However, when the dark pool is available, the uninformed chooses a DO since it offers a better price, but if θ_2^U is sufficiently low, then these ex-ante expected profits are low and, therefore, the uninformed trader is better off without dark pool. Thus, in this situation, it is not beneficial for an uninformed trader that the informed trader leaves the exchange and migrates to the dark pool.

Finally, due to the introduction of the dark pool, the expected profits of a liquidity trader in the second trading period are larger or equal if the informed trader chooses MO at $t = 1$ when the dark pool is unavailable. In contrast, in equilibria where the informed trader chooses LO at $t = 1$ when the dark is unavailable, the expected profits of a liquidity trader are lower or equal with the dark pool.

6 Empirical and Policy Implications

In this section we discuss the empirical implications of our model that can potentially be tested in applied work. The analysis that we perform in Section 5 gives us predictions of how adding a dark pool alongside an exchange affects market quality and welfare depending on stock market characteristics. In addition, our sequential model allows us to distinguish between the *initial* effects of adding a dark pool alongside an exchange ($t = 1$) from the effects that are *subsequently* present once the dark pool has been introduced ($t = 2$). These implications are relevant for the current policy and regulatory debate regarding the effects of dark trading on price informativeness, fragmentation of the order flow, market liquidity, trade creation and investors' welfare.

The existing empirical research often gives conflicting results when studying the effects of adding a dark pool alongside an exchange. Papers differ in their research questions, the type of data and regulatory environments. As a result, most of these empirical papers suggest that the discrepancies are driven by differences in the market structure and financial regulation. Interestingly, our model predicts that introducing a dark pool can have both negative and positive effects on the market performance of the LOB even if the market structure and the regulatory environment are exactly the same. As shown in Corollary 1, stock and trader characteristics affect the optimal order submission strategies and, in turn, these have implications for market quality and investors' welfare. The rest of the section uses the stock categorization in terms of volatility and information asymmetry defined in Section 3. The predictions of this section are all in relation to the benchmark where only trading in the LOB is possible.

The first set of predictions concerns price informativeness, which is at the heart of the regulatory debate about whether dark pools increase or reduce price discovery.

Prediction 1 *[Price informativeness]*
Dark pool activity leads to (ceteris paribus):

- *an initially lower price informativeness;*
- *subsequently, price informativeness might be higher or lower for all types stocks.*

Our first prediction is a consequence of the results that we obtain in relation to the segmentation of the order flow. We find that, in the first trading period, there is segmentation of the order flow (only the informed trader sends an order to the dark if conditions are favorable), while in the second trading period, there might or might not be segmentation of the order flow (since both the informed and uninformed trader might submit orders to the dark pool if conditions are favorable). The prediction that initial price informativeness falls due to the introduction of the dark pool is consistent with the existing empirical results of Hendershott and Jones (2005), Comerton-Forde and Putniņš (2015) when the proportion of dark trading is higher than 10%, and Hatheway et al. (2017). With regards to the initial segmentation of the order flow, Naes and Odegaard (2006) find that there is informational content in crossing network trades, while Nimalendran and Ray (2014) find that informed traders strategically use both crossing networks and exchanges.[23] Furthermore, our results for the subsequent trading period, when price informativeness increases due to the introduction of dark pool, are consistent with the empirical evidence of Comerton-Forde and Putniņš (2015) that show that for low levels of dark trading the effects on price discovery are benign or beneficial. For given stock characteristics, the segmentation of the order flow in the second trading period can be consistent with empirical papers that find that informed traders concentrate in the exchange while uninformed traders use the dark pool, such as Ready (2014) and Comerton-Forde and Putniņš (2015).

The second type of predictions concerns market liquidity (this can be measured by the expected inside spread in the exchange).

Prediction 2 *[Expected inside spread]*
Introducing a dark pool alongside a LOB leads to (ceteris paribus):

- *an initially and subsequently lower or equal expected inside spread for high volatility stocks;*
- *an initially higher or equal expected inside spread for low volatility stocks.*

Our theoretical results potentially reconcile the mixed empirical results previously found in the literature. Several studies that show that dark pools decrease market liquidity (Nimalendran and Ray, 2014; Weaver, 2014; Kwan et al., 2015; Degryse et al., 2015; Hatheway et al., 2017). Our model predicts that these results are relevant initially for low volatility stocks, where an informed trader supplies liquidity in the benchmark model, and the expected spread is higher when there is migration to the dark pool. Other studies show that dark pool trading increases market liquidity (Gresse, 2006; Buti et al., 2011; Ready, 2014; Aquilina et al., 2017). Our model predicts that these results will occur in high volatility stocks, where an informed trader demands liquidity in the

[23]For the purposes of this paper, crossing networks are similar to dark pools.

benchmark model, and the expected inside spread is lower when there is migration to the dark pool. Finally, Foley and Putniņš (2016) show that mid-point dark trading in Canadian market does not benefit nor harm market liquidity and Gresse (2017) shows that dark trading is not harmful to any dimension of market liquidity. In our model this occurs only initially if the execution risk in the dark pool is sufficiently high so that there is no order migration to the dark pool.

Our results are also related to the empirical work which studies how effects of dark pools depend on the tick size. In our model, the classification of high/low volatility stocks depends on the tick size, among other dimensions. Ceteris paribus, it can be seen that as the tick size increases, the low volatility region expands (where the informed trader submits a limit order when the dark pool is not available). This implies that adding a dark pool initially increases the expected inside spread if there is order migration to the dark venue. This result is similar to the one of Buti et al. (2015) that show that allowing dark orders to "queue-jump" displayed orders reduces traders' willingness to display limit orders on competing lit markets. Our results are also consistent with Buti et al. (2011) and Kwan et al. (2015) that show that when spreads on traditional exchanges are constrained by minimum pricing increments then traders have incentives to migrate towards dark venues since the execution risk in the dark is lower than the execution risk of limit orders in the exchange. Note that our results are similar, but the mechanism is different from the one explained by Buti et al. (2011) and Kwan et al. (2015). In our model the tick size does not affect the execution probability but it affects the profits obtained in case of execution.

The third set of predictions concerns expected volume and trade creation or destruction.

Prediction 3 *[Expected Volume and Trade Creation]*
Introducing a dark pool alongside a LOB leads to the following effects on:
(i) expected trading volume in the exchange. Ceteris paribus,

- *for high volatility stocks, it is initially lower when there is order migration to the dark pool, and subsequently it is lower or equal;*
- *for low volatility stocks, it does not change initially even if there is order migration to the dark venue, and subsequently it is lower or equal.*

(ii) trade creation or destruction (total expected trading volume). If there is order migration to the dark venue, ceteris paribus,

- *there is initially trade destruction for high volatility stocks and trade creation for low volatility stocks.*

Our third prediction is related to studies that analyze how stock heterogeneity affects dark pool trading. Gresse (2006) shows that crossing networks do not attract orders from illiquid stocks, but they attract orders on stocks that are infrequently traded in the exchange, thus suggesting the possibility that crossing networks might foster trade creation. Furthermore, Menkveld et al. (2017)

shows that their pecking order theory (on top of the pecking order are the low-cost, low-immediacy venues such as midpoint dark pools, whereas at the bottom are the high-cost, high-immediacy venues such as lit exchanges) varies with stock characteristics (size). This can be explained in our model by the cut-off execution probability in the dark that generates order migration, which is lower for low volatility, low liquidity stocks and becomes larger for higher volatility, higher liquidity stocks (see Figure 5). Consequently, traders preference for the lit market is higher for high volatility or high liquidity stocks. Thus our results are consistent with Menkveld et al. (2017) since they show that the pecking order pattern is weaker if the spread is lower. Similarly, Degryse et al. (2018) show that the higher the stock volatility, the lower the volume of dark trading is. They also study how stock liquidity affects dark trading and obtain that visible depth and the quoted spread do not significantly affect dark trading.

The fourth set of predictions concerns the expected profits of each type of trader.

Prediction 4 *[Investors' welfare: expected profits]*

Introducing a dark pool alongside a LOB leads to the following effects for each type of trader (ceteris paribus):

- *Informed traders: initially and subsequently the profits are strictly larger if there is order migration to the dark venue.*
- *Uninformed traders: initially profits are larger or equal.*
- *Liquidity traders: initially profits remain unchanged and subsequently, if there is order migration to the dark, then profits are strictly larger for high volatility stocks and are strictly lower for low volatility stocks.*

Our results that the profits of informed traders are strictly larger if there is order migration to the dark venue can be consistent with the results found in the experimental paper of Bloomfield et al. (2015) when informed traders have very valuable private information. To the best of our knowledge, there are not further empirical papers that are testing the effects of introducing a dark pool alongside an exchange on traders' profits.

Furthermore, our model can inform the regulatory debate on dark pools. We use our model to formulate two additional policy related predictions. First, the European Commission aims to limit dark trading through the Double Volume Cap (DVC) mechanism as part of MiFID II/MiFIR, which is being implemented since 2018. The DVC introduces a cap on dark trading that limits the trading volume of a financial instrument in any single dark pool to 4% of its total volume of trading volume in the previous year. This cap on dark trading can be thought to be equivalent to an upper limit on the execution probability in the dark. If the cap on dark trading is binding so that the informed trader is restricted to trading in the exchange instead of trading in the dark pool in the first trading period, then the DVC limits the benefits but also the drawbacks of introducing a dark pool alongside an exchange. The next prediction concerns the effect of adding a cap on dark trading relative to the case when there is no restriction on dark trading.

Prediction 5 *[Effects of a Cap on Dark Trading]*
Imposing a binding cap on dark trading has the following initial effects on market quality:

- *price informativeness will be higher;*
- *the expected inside spread will be higher for high volatility stocks and lower for low volatility stocks;*
- *trading volume will be higher for high volatility stocks and will remain the same as without the cap for low volatility stocks; there will be trade creation for high volatility stocks and trade destruction for low volatility stocks;*
- *the profits for informed and uninformed traders will not be higher, while for liquidity traders profits will remain unchanged.*

Our prediction shows that while the DVC is expected to have an initial positive effect on private informativeness, its effects on the other market quality parameters depend on stock characteristics. Specifically, the DVC policy might have unintended negative consequences, such as decreasing liquidity for high volatility stocks or generating trade destruction for low volatility stocks, or decreasing the profits for rational traders.

Finally, we address one of the very controversial restrictions on dark pools related to high frequency traders (HFT). Many dark pools advertise that they provide investors with protection against HFT predatory trading.[24] HTF traders have a speed and technology advantage in relation to other traders, and as such, they can trade ahead other traders. We assume that informed traders in our model may be considered to be HFT traders since technology helps them to access and process information faster. Therefore, our framework allows us to understand the impact of limiting the access of HFT to the dark pool on investors' welfare. Restricting the access of informed traders to the dark pool will lead to exactly the same equilibria as in the case when the probability of the execution for the informed trader in the dark pool in the first trading period is very small, so that the informed trader remains in the exchange. Hence, both the benefits and drawbacks of introducing a dark pools will disappear if such restriction is imposed. For the following prediction, we concentrate on investors' welfare.

Prediction 6 *[Effects of restricting HFT trading in dark pool on investors' welfare]*
Limiting HFT trading in dark pool initially does not increase the profits of an uninformed trader, and subsequently it decreases both the profits of uninformed and liquidity traders for the "High-Low" stocks.

Our policy predictions show that policymarkers need to be aware that the effects of imposing such restrictions depend on stock characteristics, trading period, and market quality/welfare indicator.

[24]See NY Attorney General v Barclays case for a case of misconduct by Barclays dark pool. US SEC Commissioner Luis Aguilar (2015) remarked that "dark pools initially portrayed themselves as havens from predatory traders. They achieved this, in part, by excluding high frequency traders, who supposedly use brute speed to front-run institutional investors' large orders. Lured by this promise of safety, institutional traders embraced ATSs as a solution to their trading needs. Unfortunately, all too often the safety these investors sought proved illusory."

7 Concluding Remarks

This paper studies the impact of introducing an opaque dark pool alongside a transparent exchange organized as a limit order book in a sequential model with asymmetric information about the value of the asset. In the initial trading period, we find that only the informed trader diverts his order from the exchange to the dark pool provided that the execution risk in the dark pool is sufficiently low. However, in the succeeding trading period, both informed and uninformed traders may trade in the dark pool if conditions are favorable. Our paper shows how the introduction of a dark pool affects market quality and investors' welfare. In terms of market quality, we find that it is crucially determined by whether the order that migrates to the dark venue comes from an informed or an uninformed trader that was demanding or supply liquidity, or not participating in the market when the dark pool was unavailable. Given that in the first trading period only the order of an informed trader might migrate to the dark pool, we find that the informational content of prices is never initially enhanced by the addition of a dark pool. However, in the second trading period, this result might not hold because both informed and uninformed traders might choose to trade in the dark venue. With regards to market liquidity, we find that if the informed trader demands liquidity in the first trading period, then the expected inside spread in the exchange is not higher with the introduction of the dark pool. Otherwise, market liquidity might decrease in the first or second trading periods. Furthermore, we find that the addition of the dark pool might produce trade creation or destruction depending on stock and trader characteristics. Concerning investors' welfare, we find that expected profits in the first trading period are not lower for all types of traders with the introduction of the dark pool. However, this result might not hold in the second trading period since expected profits of uninformed and liquidity traders may be reduced because of a deterioration of market quality.

Our policy analysis concludes that regulators should take into account that establishing measures to reduce informed traders' participation in dark pools could negatively affect other investors and some market quality indicators.

Future work could extend our theoretical model in different ways, such as in considering that orders can be of different sizes, or in assuming that prices are asymmetrically located in the grid, or in examining a dark pool where the price improvement is different from the midpoint of the spread. Additionally, our results call for more empirical and experimental work which test the predictions of our model, and more generally, for the development of applied work which studies the effects of asymmetric information in the competition between trading venues with different degrees of transparency on market quality and investors' welfare.

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Appendices

A Benchmark model without dark pool

Proof of Lemma 1. We define Ω_o and Γ_o as the probability that an informed trader and uninformed trader at $t=1$ choose an order $\mathcal{O} \in \mathbb{O}_{ND}$, where $o=0$ corresponds to a NT order; $o=1$ to a MO; $o=2$ to a LO; and such that $\sum_{o=0}^{2} \Omega_o = 1$, and $\sum_{o=0}^{2} \Gamma_o = 1$. We also define by $\mathbb{B}$ the set of all possible states of the LOB at the end of the first trading period and by $\mathcal{B}_1 \in \mathbb{B}$ a possible state of the book such that

$$\mathcal{B}_1 = \begin{cases} \varnothing, & \text{if the best prices in the book are } (A_1^1, B_1^1) \\ BMO, & \text{if the best prices in the book are } (A_1^2, B_1^1) \\ BLO, & \text{if the best prices in the book are } (A_1^1, B_1^1+\tau) \\ SMO, & \text{if the best prices in the book are } (A_1^1, B_1^2) \\ SLO, & \text{if the best prices in the book are } (A_1^1-\tau, B_1^1). \end{cases}$$

Note that the state of the book $\mathcal{B}_1=\varnothing$, can be obtained either because no trader arrived or a trader arrived but he decided not to trade, while the other states of the book are uniquely determined by the traders' actions at $t=1$.

We solve the game backwards. At $t=2$, the expected profits for an informed and uninformed buyer and seller are:

	IH			IL		
	BMO	BLO	NT	SMO	SLO	NT
(A_1^1, B_1^1)	$(\kappa-k_1)\,\tau$	0	0	$(\kappa-k_1)\,\tau$	0	0
(A_1^2, B_1^1)	$(\kappa-k_2)\,\tau$	0	0	$(\kappa-k_1)\,\tau$	0	0
$(A_1^1, B_1^1+\tau)$	$(\kappa-k_1)\,\tau$	0	0	$(\kappa-k_1+1)\,\tau$	0	0
(A_1^1, B_1^2)	$(\kappa-k_1)\,\tau$	0	0	$(\kappa-k_2)\,\tau$	0	0
$(A_1^1-\tau, B_1^1)$	$(\kappa-k_1+1)\,\tau$	0	0	$(\kappa-k_1)\,\tau$	0	0

Table A.1: Expected profits of an informed buyer (IH) and an informed seller (IL) at $t=2$ when traders do not have access to the dark pool.

	UB			US		
	BMO	BLO	NT	SMO	SLO	NT
(A_1^1, B_1^1)	$-k_1\tau$	0	0	$-k_1\tau$	0	0
(A_1^2, B_1^1)	$(X\kappa-k_2)\,\tau$	0	0	$-(k_1+X\kappa)\,\tau$	0	0
$(A_1^1, B_1^1+\tau)$	$(Y\kappa-k_1)\,\tau$	0	0	$-(k_1-1+Y\kappa)\,\tau$	0	0
(A_1^1, B_1^2)	$-(X\kappa+k_1)\,\tau$	0	0	$(X\kappa-k_2)\,\tau$	0	0
$(A_1^1-\tau, B_1^1)$	$-(Y\kappa+k_1-1)\,\tau$	0	0	$(Y\kappa-k_1)\,\tau$	0	0

Table A.2: Expected profits of an uninformed buyer (UB) and an uninformed seller (US) at $t=2$ when traders do not have access to the dark pool.

Note that at $t = 2$ the expected profits of each strategy depend on the state of the LOB (which on its turn depends on the chosen strategy at $t = 1$). Uniformed traders at $t = 2$ form beliefs about the strategies and type of player in $t = 1$. Thus, we define the uninformed traders' belief at $t = 2$ about the probability that the MO observed in the LOB was submitted by an informed trader as

$$X = \frac{\lambda \pi \Omega_1}{1 - \lambda + \lambda \pi \Omega_1 + \lambda (1 - \pi) \Gamma_1}. \tag{A.1}$$

Similarly, we define the uninformed traders' belief at $t = 2$ about the probability that the LO (observed in the LOB) was submitted by an informed trader as

$$Y = \frac{\pi \Omega_2}{\pi \Omega_2 + (1 - \pi) \Gamma_2}. \tag{A.2}$$

By comparing the expected profits of an informed trader in $t = 2$ we obtain that the informed trader always submits MO. Similarly, we compare the profits of an uniformed trader and see that he never chooses to submit a LO. Their choice between MO or NT depends on the uninformed trader beliefs that the order placed at $t = 1$ that he observes in the book comes from an informed trader, as it can be seen in *Table A.3.*

State of the Book	*UB*	*US*
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } X\kappa > k_2 \\ NT & \text{if } X\kappa \leq k_2 \end{cases}$	NT
$(A_1^1, B_1^1 + \tau)$	$\begin{cases} MO & \text{if } Y\kappa > k_1 \\ NT & \text{if } Y\kappa \leq k_1 \end{cases}$	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } X\kappa > k_2 \\ NT & \text{if } X\kappa \leq k_2 \end{cases}$
$(A_1^1 - \tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } Y\kappa > k_1 \\ NT & \text{if } Y\kappa \leq k_1 \end{cases}$

Table A.3: Optimal trading strategies of an uniformed buyer (UB) and seller (US) at $t = 2$ when traders do not have access to the dark pool.

At $t = 1$, the expected profits of an informed and uniformed trader are presented in Table $A.4$ and Table $A.5$, respectively. It can be easily seen that the informed trader never chooses NT, while the uninformed never chooses MO.

IH	IL	$Expected\ Profits$
BMO	SMO	$(\kappa - k_1)\,\tau$
BLO	SLO	$\delta\frac{1-\lambda}{2}\,(\kappa + k_1 - 1)\,\tau$
NT	NT	0

Table A.4: Expected profits of an informed buyer (IH) and seller (IL) at $t = 1$ when traders do not have access to the dark pool.

UB	US	$Expected\ Profits$
BMO	SMO	$-k_1\tau$
BLO	SLO	$\frac{\delta}{2}\,((1-\lambda+\lambda\pi)\,(k_1-1) - \lambda\pi\kappa)\,\tau$
NT	NT	0

Table A.5: Expected profits of an uninformed buyer (UB) and seller (US) at $t = 1$ when traders do not have access to the dark pool.

■

Proof of Proposition 1. The procedure we follow to check if a particular strategy profile constitutes a PBE is as follows:

1. Specify a strategy profile for rational traders at $t = 1$.
2. Update the beliefs of the uninformed trader at $t = 2$ using Bayes' rule at all information sets, whenever possible.
3. Given their beliefs, find the optimal response for the traders at $t = 2$.
4. Given the optimal response of traders at $t = 2$, and using Tables A.4 and A.5 find the optimal action for rational traders at $t = 1$.
5. Check if the optimal strategy profile for the traders at $t = 1$ coincide with the profile suggested in step 1.

We apply the procedure outlined above to check when each possible strategy profile can be an equilibrium. Because of the symmetry of the model, without any loss of generality, at $t = 1$ we focus on buyers. We present the proof for one of the possible strategy profile at $t = 1$ that yields an equilibrium. The proofs of all the other 3 equilibria can be obtained on request from the authors.

$\mathcal{E}_1^{ND}$**:** (BMO, SMO, BLO, SLO)
First step. In this case $\Omega_0 = 0$, $\Omega_1 = 1$, $\Omega_2 = 0$, $\Gamma_0 = 0$, $\Gamma_1 = 0$, and $\Gamma_2 = 1$.
Second step. Using Bayes' rule we obtain that $X = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$ and $Y = 0$.
Third step. Applying Lemma 1, we know that at $t = 2$ the optimal strategy of informed traders is to choose a MO, while the optimal strategy of the uninformed trader is as follows:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$	NT
$(A_1^1, B_1^1+\tau)$	NT	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$
$(A_1^1-\tau, B_1^1)$	NT	NT

Table A.6: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO).

Fourth step. Given the optimal response of traders at $t = 2$, we find the optimal action for all rational traders at $t = 1$.

Informed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$(\kappa - k_1)\,\tau \geq \delta\frac{1-\lambda}{2}\,(\kappa + k_1 - 1)\,\tau. \tag{A.3}$$

Uninformed traders at $t = 1$ have no incentives incentive to deviate from the prescribed strategy if and only if

$$(1-\lambda)\,(k_1 - 1) - \lambda\pi\,(\kappa - (k_1 - 1)) > 0. \tag{A.4}$$

Fifth step. Nobody at $t = 1$ has unilateral incentives to deviate from (BMO, SMO, BLO, SLO) when both conditions $(A.3)$ and $(A.4)$ are satisfied, and these conditions can be rewritten as

$$\kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN < \psi^U_{LO-NT}, \tag{A.5}$$

where the expression of ψ^U_{LO-NT} and κ^I_{MO-LO} are given in the statement of this proposition.

$\mathcal{E}_2^{ND}$**:** (BMO, SMO, NT, NT)

Following the same procedure we obtain that in this case nobody at $t = 1$ has unilateral incentives to deviate whenever:

$$\kappa - k_1 \;\geq\; \delta\frac{1-\lambda}{2}\,(\kappa + k_1 - 1) \tag{A.6}$$

$$0 \;\geq\; (1-\lambda+\lambda\pi)\,(k_1 - 1) - \lambda\pi\kappa. \tag{A.7}$$

which can be rewritten as $\kappa^I_{MO-LO}\tau \leq \sigma$ and $PIN \geq \psi^U_{LO-NT}$.

Applying Lemma 1, we know that at $t = 2$ the optimal strategy of informed traders is to choose a MO, while the optimal strategy of uninformed trader is:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$	NT
$(A_1^1, B_1^1+\tau)$	$\begin{cases} MO & \text{if } Y\kappa > k_1 \\ NT & \text{if } Y\kappa \leq k_1 \end{cases}$	NT
(A_1^1, B_1^2)	NT	$\begin{cases} MO & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa > k_2 \\ NT & \text{if } \frac{\lambda\pi}{1-\lambda+\lambda\pi}\kappa \leq k_2 \end{cases}$
$(A_1^1-\tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } Y\kappa > k_1 \\ NT & \text{if } Y\kappa \leq k_1 \end{cases}$

Table A.7: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, NT, NT).

$\mathcal{E}_3^{ND}$**:** (BLO, SLO, BLO, BLO)

Following the same procedure we obtain that in this case nobody at $t = 1$ has unilateral incentives to deviate whenever:

$$\delta\frac{1-\lambda}{2}(\kappa + k_1 - 1) \;>\; \kappa - k_1 \tag{A.8}$$

$$(1-\lambda+\lambda\pi)(k_1 - 1) - \lambda\pi\kappa \;>\; 0, \tag{A.9}$$

which can be rewritten as $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$.

Applying Lemma 1, we know that at $t = 2$ the optimal strategy of informed traders is to choose a MO, while the optimal strategy of uninformed trader is:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	NT
$(A_1^1, B_1^1+\tau)$	$\begin{cases} MO & \text{if } \pi\kappa > k_1 \\ NT & \text{if } \pi\kappa \leq k_1 \end{cases}$	NT
(A_1^1, B_1^2)	NT	NT
$(A_1^1-\tau, B_1^1)$	NT	$\begin{cases} MO & \text{if } \pi\kappa > k_1 \\ NT & \text{if } \pi\kappa \leq k_1 \end{cases}$

Table A.8: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, BLO, BLO).

$\mathcal{E}_4^{ND}$: (BLO, SLO, NT, NT)

Following the same procedure we obtain that in this case nobody at $t = 1$ has unilateral incentives to deviate whenever:

$$\delta \frac{1-\lambda}{2} (\kappa + k_1 - 1) \tau \quad > \quad (\kappa - k_1) \tau \tag{A.10}$$

$$0 \quad \geq \quad \frac{\delta}{2} \left((1-\lambda)(k_1 - 1) - \lambda \pi (\kappa - (k_1 - 1)) \right). \tag{A.11}$$

which can be rewritten as $\sigma < \kappa^I_{MO-LO} \tau$ and $PIN \geq \psi^U_{LO-NT}$.

Applying Lemma 1, we know that at $t = 2$ the optimal strategy of informed traders is to choose a MO, while the optimal strategy of uninformed trader is:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	NT
$(A_1^1, B_1^1 + \tau)$	MO	NT
(A_1^1, B_1^2)	NT	NT
$(A_1^1 - \tau, B_1^1)$	NT	MO

Table A.9: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, NT, NT).

■

Proposition 7 *If $k_1 = 1$, then a PBE of the game is as follows: (BMO, SMO, NT, NT) is the optimal strategy profile for traders at $t = 1$. The beliefs of uninformed traders at $t = 2$ are $X = \frac{\lambda\pi}{1 - \lambda + \lambda\pi}$ and $Y = p \in [0, 1]$. The optimal strategy of informed traders at $t = 2$ is to choose MO for all possible states of the book and the optimal strategies of uninformed traders at $t = 2$ are described in Table A.7.*

Proof of Proposition 7. The full proof of this Proposition is omitted since we have to replace $k_1 = 1$ in the proof of Proposition 1. It should only be noted that when $k_1 = 1$ the conditions $(A.4)$ and $(A.9)$ are never satisfied and, therefore, the strategies (BMO, SMO, BLO, SLO) and (BLO, SLO, BLO, BLO) cannot be part of an equilibrium of the game. By contrast, when $k_1 = 1$, the conditions $(A.7)$ and $(A.11)$ are always satisfied. However the condition $(A.10)$ is never satisfied when $k_1 = 1$ and, therefore, the strategy (BLO, SLO, NT, NT) cannot be either part of an equilibrium of the game. ■

B Model with dark pool

Definition 1 *Let us define, similarly to the model without a dark pool,* Ω_3 *and* Γ_3 *as the probability that an informed trader and uninformed trader at* $t=1$ *choose a DO, and such that* $\sum_{o=0}^{3}\Omega_o = 1$, *and* $\sum_{o=0}^{3}\Gamma_o = 1$.

Proof of Lemma 2. Applying Bayes' rule, it follows that

$$\begin{aligned}\theta_2^I &= \lambda\left(\pi\Omega_3 + \frac{(1-\pi)\,\Gamma_3}{2}\right)\theta_1^{I,2} + \left(1-\lambda\left(\pi\Omega_3 + \frac{(1-\pi)\,\Gamma_3}{2}\right)\right)\theta_1^{I,1} \text{ and} \\ \theta_2^U &= \lambda\left(\frac{\pi\Omega_3}{2} + \frac{(1-\pi)\,\Gamma_3}{2}\right)\theta_1^{U,2} + \left(1-\lambda\left(\frac{\pi\Omega_3}{2} + \frac{(1-\pi)\,\Gamma_3}{2}\right)\right)\theta_1^{U,1}.\end{aligned}$$

Hence, we have that if the probabilities of execution of dark orders at $t=1$ coincide for all rational traders ($\theta_1^{I,2}=\theta_1^{U,2}$ and $\theta_1^{I,1}=\theta_1^{U,1}$), then $\theta_2^U-\theta_2^I=\Omega_3\frac{\lambda\pi}{2}(\theta_1^{I,1}-\theta_1^{I,2})\geq 0$ (with strict inequality when $\Omega_3\neq 0$). ■

Proof of Lemma 3. Note that in the case there is a dark pool the set of the possible states of the LOB is the same as in the case there is no dark pool. However, the state of the book $\mathcal{B}_1$=∅, can be obtained either because a trader arrived and decided not to trade, or because a trader arrived and he submitted a *DO*.

We solve the model backwards. At $t=2$ the expected profits of each strategy depend on the state of the LOB. Additionally, uninformed traders form beliefs about the strategies that have been chosen at $t=1$. Let X and Y be defined as in (A.1) and (A.2), respectively, and Z denote the uninformed trader's belief at $t=2$ about the probability that a *DO* was submitted by an informed, which is equal to

$$Z=\frac{\pi\Omega_3}{\pi\Omega_3+(1-\pi)\,\Gamma_3}. \tag{B.1}$$

As in the case when the dark pool was not available, and without loss of generality, we will focus on the expected profits for an informed and an uniformed buyer at $t=2$, as summarized below. The expected profits of an informed buyer at $t=2$ are

IH	BMO	BDO	BLO	NT
(A_1^1, B_1^1)	$(\kappa - k_1)\,\tau$	$\theta_2^I \kappa\tau$	$P_I \delta\,(k_1 + \kappa - 1)\,\tau$	0
(A_1^2, B_1^1)	$(\kappa - k_2)\,\tau$	$\theta_2^I \left(\kappa - \frac{k_2 - k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(\kappa - k_1)\,\tau$	$\theta_2^I \left(\kappa - \frac{1}{2}\right)\tau$	0	0
(A_1^1, B_1^2)	$(\kappa - k_1)\,\tau$	$\theta_2^I \left(\kappa + \frac{k_2 - k_1}{2}\right)\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$(\kappa - k_1 + 1)\,\tau$	$\theta_2^I \left(\kappa + \frac{1}{2}\right)\tau$	0	0

Table B.1: Expected profits of an informed buyer (IH) at $t = 2$

where P_I is the probability of execution of a limit order placed by an informed trader at $t = 2$ conditional on the fact that there is no change in the LOB during the first trading period, and equals

$$P_I = p_{BLO,2}^{IH}\,(\mathcal{B}_1 = \varnothing) = \frac{(1 - \theta_1^U)\frac{1-\pi}{2}\Gamma_3}{\pi\Omega_3 + (1-\pi)(\Gamma_0 + \Gamma_3)}.$$

Similarly, the expected profits of an uninformed buyer at $t = 2$ are summarized as follows:

UB	BMO	BDO	BLO	NT
(A_1^1, B_1^1)	$-k_1\tau$	0	$P_U \delta\,(k_1 - Z\kappa - 1)\,\tau$	0
(A_1^2, B_1^1)	$(X\kappa - k_2)\,\tau$	$\theta_2^U \left(X\kappa - \frac{k_2 - k_1}{2}\right)\tau$	0	0
$(A_1^1, B_1^1 + \tau)$	$(Y\kappa - k_1)\,\tau$	$\theta_2^U \left(Y\kappa - \frac{1}{2}\right)\tau$	0	0
(A_1^1, B_1^2)	$-\,(X\kappa + k_1)\,\tau$	$-\theta_2^U \left(X\kappa - \frac{k_2 - k_1}{2}\right)\tau$	0	0
$(A_1^1 - \tau, B_1^1)$	$-\,(Y\kappa + k_1 - 1)\,\tau$	$-\theta_2^U \left(Y\kappa - \frac{1}{2}\right)\tau$	0	0

Table B.2: Expected profits of an uninformed buyer (UB) at $t = 2$

where P_U is the probability of execution of a limit order placed by an uninformed trader at $t = 2$ given that there are no changes in prices in the LOB during the first trading period, and equals

$$P_U = p_{BLO,2}^{UB}\,(\mathcal{B}_1 = \varnothing) = \frac{1}{2}\frac{(1 - \theta_1^I)\pi\Omega_3 + \left(1 - \theta_1^U\right)(1-\pi)\Gamma_3}{\pi\Omega_3 + (1-\pi)(\Gamma_0 + \Gamma_3))}.$$

At $t = 1$ the expected profits of an informed IH and an uniformed buyer UB are summarized in Table *B.3* and Table *B.4*, respectively.[25]

[25]Notice that due to the symmetry of the game, the expected profits of the informed IL trader and uninformed seller US are the same as the ones displayed in Tables *B.3* and *B.4*, respectively.

IH	*Expected Profits at* $t = 1$
BMO	$(\kappa - k_1)\,\tau$
BLO	$\dfrac{(1-\lambda)\delta}{2}\,(\kappa + k_1 - 1)\,\tau$
BDO	$\theta_1^I \kappa\tau + (1-\theta_1^I)\delta^2 \left(\lambda \frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} + \right.$ $\left. +(\kappa - k_1) - (k_2 - k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right)$
NT	0

Table B.3: Expected profits of an informed buyer (IH) at $t = 1$

UB	*Expected Profits at* $t = 1$
BMO	$-k_1\tau$
BLO	$\dfrac{\delta}{2}\left((1-\lambda)(k_1 - 1) - \lambda\pi I_{SMO,2}^{IL,\mathcal{B}_1=BLO}(\kappa - k_1 - 1)\right)\tau$
BDO	$(1-\theta_1^U)\delta^2\left(\frac{\lambda}{2}(\pi I_{SLO,2}^{IL,\mathcal{B}_1=\varnothing} + (1-\pi) I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}) + \right.$ $\left. +(\frac{\lambda\pi}{2} I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2})(k_1 - k_2) - k_1\right)\tau$
NT	0

Table B.4: Expected profits of an uninformed buyer (UB) at $t = 1$

where $I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}$, and $I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing}$ are indicator functions such that $I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} = 1$ if at $t = 2$, an US selects a SLO when the LOB has not changed at $t = 1$, and $I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} = 0$, otherwise. Similarly, the remainder indicator functions can be defined. By simply inspection of the payoff in Table B.3 can be seen that the informed buyers at $t = 1$ never choose NT because this order is dominated by placing a MO.

Notice also that the expected profits of a BDO submitted by an informed buyer at $t = 1$ may be positive, and as a result the informed may choose to place a BDO at $t = 1$ depending on how high is the execution probability θ_1^I. However, the payoff at $t = 1$ of the BDO for the uninformed trader is always negative. To see this we rewrite this payoff as

$$\theta_1^U \cdot 0 + (1-\theta_1^U)\delta^2\left(-k_1\tau + \frac{\lambda}{2}\left(\pi I_{SLO,2}^{IL,\mathcal{B}_1=\varnothing} + (1-\pi) I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}\right)\tau - \left(\frac{\lambda\pi}{2} I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)(k_2 - k_1)\,\tau\right). \tag{B.2}$$

This is because if a BDO is executed at $t = 1$, then its expected profits are zero, which occurs with probability θ. If the order is not executed at $t = 1$ and returns to the market at $t = 3$, which occurs with probability $1-\theta$, then expected profits depend on whether the uninformed trader who returns to the exchange decides to submit a NT or MO. If the uninformed trader selects NT then the expected profits equal zero. If he submits a MO then the profit consists of three terms. The first consists of the expected profits of a BMO at $t = 1$ for an UB (i.e., $-k_1\tau$). The second is the potential increase in profits due to the possibility that at $t = 2$ a new trader arrives and submits a SLO leading to a better price for the uninformed buyer (given by $\frac{\lambda}{2}\left(\pi I_{SLO,2}^{IL,\mathcal{B}_1=\varnothing} + (1-\pi) I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}\right)\tau$). The third is the potential decrease in profits due to the possibility that a trader at $t = 2$ submits a BMO and, consequently, the MO that arrives at $t = 3$ from the dark pool is executed at a worse price (given by $(-\left(\frac{\lambda}{2}\pi I^{IH,\mathcal{B}_1=\varnothing}_{BMO,2} + \frac{1-\lambda}{2}\right)(k_2 - k_1)\,\tau)$. We find that the increase in expected profits due to the potential arrival of a SLO at $t = 2$ is not greater than $-k_1\tau$ and these losses might be even greater in case that a BMO is submitted at $t = 2$. Consequently, an uninformed trader will never choose a DO at $t = 1$. As a result, an uninformed trader never selects a DO and therefore $\Gamma_3 = 0$ which implies

$$P_I = p^{IH}_{BLO,2}\,(\mathcal{B}_1 = \varnothing) = p^{IL}_{SLO,2}\,(\mathcal{B}_1 = \varnothing) = 0.$$

Consequently, informed traders never choose a LO at $t = 2$, since this order is also dominated by a MO.

Let us determine next the optimal strategy for each trader. Depending on the values of the parameters we have 6 possible cases for the informed trader and 16 cases for the uninformed trader.

First let us first focus on the informed traders. Note that since $\kappa > k_2 > k_1 \geq 1$, the following inequalities hold

$$\frac{\kappa - k_2}{\kappa - \frac{k_2-k_1}{2}} < \frac{\kappa - k_1}{\kappa + \frac{k_2-k_1}{2}} < \frac{\kappa - k_1}{\kappa} < \frac{\kappa - k_1}{\kappa - \frac{1}{2}} < \frac{\kappa - k_1 + 1}{\kappa + \frac{1}{2}}.$$

We define by

$$\begin{aligned} \theta_X &\equiv \frac{X\kappa - k_2}{X\kappa - \frac{k_2-k_1}{2}} \\ \theta_Y &\equiv \frac{Y\kappa - k_1}{Y\kappa - \frac{1}{2}} \end{aligned}$$

$$\begin{aligned} BX &= \begin{cases} BMO \text{ if } p^{IH,\mathcal{B}_1=\varnothing}_{BLO,2} \leq \frac{\kappa-k_1}{\delta(\kappa+k_1-1)} \\ BLO \text{ if } p^{IH,\mathcal{B}_1=\varnothing}_{BLO,2} > \frac{\kappa-k_1}{\delta(\kappa+k_1-1)}, \end{cases} \\ SX &= \begin{cases} SMO \text{ if } p^{IL,\mathcal{B}_1=\varnothing}_{SLO,2} \leq \frac{\kappa-k_1}{\delta(\kappa+k_1-1)} \\ SLO \text{ if } p^{IL,\mathcal{B}_1=\varnothing}_{SLO,2} > \frac{\kappa-k_1}{\delta(\kappa+k_1-1)}, \end{cases} \\ BY &= \begin{cases} BDO \text{ if } p^{IH,\mathcal{B}_1=\varnothing}_{BLO,2} < \frac{\theta^I_2\kappa}{\delta(\kappa+k_1-1)} \\ BLO \text{ if } p^{IH,\mathcal{B}_1=\varnothing}_{BLO,2} \geq \frac{\theta^I_2\kappa}{\delta(\kappa+k_1-1)}, \end{cases} \\ SY &= \begin{cases} SDO \text{ if } p^{IH,\mathcal{B}_1=\varnothing}_{BLO,2} < \frac{\theta^I_2\kappa}{\delta(\kappa+k_1-1)} \\ SLO \text{ if } p^{IH,\mathcal{B}_1=\varnothing}_{BLO,2} \geq \frac{\theta^I_2\kappa}{\delta(\kappa+k_1-1)}. \end{cases} \end{aligned}$$

The optimal strategies of the informed traders at $t = 2$ are given in Table B.5.

Condition	Optimal Strategies of Informed Traders at t = 2		
	State of the Book	**IH**	**IL**
Case I_1 $\theta_2^I \leq \frac{\kappa-k_2}{\kappa-\frac{k_2-k_1}{2}}$	(A_1^1, B_1^1)	BX	SX
	(A_1^2, B_1^1)	BMO	SMO
	$(A_1^1, B_1^1+\tau)$	BMO	SMO
	(A_1^1, B_1^2)	BMO	SMO
	$(A_1^1-\tau, B_1^1)$	BMO	SMO
Case I_2 $\frac{\kappa-k_2}{\kappa-\frac{k_2-k_1}{2}} < \theta_2^I \leq \frac{\kappa-k_1}{\kappa+\frac{k_2-k_1}{2}}$	(A_1^1, B_1^1)	BX	SX
	(A_1^2, B_1^1)	BDO	SMO
	$(A_1^1, B_1^1+\tau)$	BMO	SMO
	(A_1^1, B_1^2)	BMO	SDO
	$(A_1^1-\tau, B_1^1)$	BMO	SMO
Case I_3 $\frac{\kappa-k_1}{\kappa+\frac{k_2-k_1}{2}} < \theta_2^I \leq \frac{\kappa-k_1}{\kappa}$	(A_1^1, B_1^1)	BX	SX
	(A_1^2, B_1^1)	BDO	SDO
	$(A_1^1, B_1^1+\tau)$	BMO	SMO
	(A_1^1, B_1^2)	BDO	SDO
	$(A_1^1-\tau, B_1^1)$	BMO	SMO
Case I_4 $\frac{\kappa-k_1}{\kappa} < \theta_2^I \leq \frac{\kappa-k_1}{\kappa-\frac{1}{2}}$	(A_1^1, B_1^1)	BY	SY
	(A_1^2, B_1^1)	BDO	SDO
	$(A_1^1, B_1^1+\tau)$	BMO	SMO
	(A_1^1, B_1^2)	BDO	SDO
	$(A_1^1-\tau, B_1^1)$	BMO	SMO
Case I_5 $\frac{\kappa-k_1}{\kappa-\frac{1}{2}} < \theta_2^I \leq \frac{\kappa-k_1+1}{\kappa+\frac{1}{2}}$	(A_1^1, B_1^1)	BY	SY
	(A_1^2, B_1^1)	BDO	SDO
	$(A_1^1, B_1^1+\tau)$	BDO	SMO
	(A_1^1, B_1^2)	BDO	SDO
	$(A_1^1-\tau, B_1^1)$	BMO	SDO
Case I_6 $\frac{\kappa-k_1+1}{\kappa+\frac{1}{2}} < \theta_2^I$	(A_1^1, B_1^1)	BY	SY
	(A_1^2, B_1^1)	BDO	SDO
	$(A_1^1, B_1^1+\tau)$	BDO	SDO
	(A_1^1, B_1^2)	BDO	SDO
	$(A_1^1-\tau, B_1^1)$	BDO	SDO

Table B.5: Optimal Strategies of Informed Traders at $t = 2$

The optimal strategies of uninformed traders at $t=2$ will be specified in each equilibria (see proof of Lemma 4, below). In equilibrium we show that neither informed nor uninformed submit LO at $t=2$. ■

Definition 2 *Let us consider next the following cut-off definitions*

$$\begin{aligned}
\underline{\theta} &\equiv \frac{\kappa-k_1}{\kappa}, \\
\overline{\theta} &\equiv \frac{\kappa-k_1+1}{\kappa+\frac{1}{2}}, \\
\widehat{\theta}_{MO-DO} &\equiv \frac{\kappa-k_1-\delta^2\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing}+\frac{1-\lambda}{2}\right)\right)}{\kappa-\delta^2\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing}+\frac{1-\lambda}{2}\right)\right)}, \\
\overline{\theta}_{MO-DO} &\equiv \frac{\kappa-k_1-\delta^2\left(\kappa-k_1-(k_2-k_1)\left(\lambda\pi+\frac{1-\lambda}{2}\right)\right)}{\kappa-\delta^2\left(\kappa-k_1-(k_2-k_1)\left(\lambda\pi+\frac{1-\lambda}{2}\right)\right)}, \\
\widehat{\theta}_{LO-DO} &\equiv \frac{\delta\frac{1-\lambda}{2}(\kappa+k_1-1)-\delta^2\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing}+\frac{1-\lambda}{2}\right)\right)}{\kappa-\delta^2\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}-(k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing}+\frac{1-\lambda}{2}\right)\right)}, \\
\underline{\theta}_{LO-DO} &\equiv \frac{\delta\frac{1-\lambda}{2}(\kappa+k_1-1)-\delta^2\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}-(k_2-k_1)\frac{1-\lambda}{2}\right)}{\kappa-\delta^2\left(\kappa-k_1+\lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing}-(k_2-k_1)\frac{1-\lambda}{2}\right)}, \\
\overline{\theta}_{LO-DO} &\equiv \frac{\delta\frac{1-\lambda}{2}(\kappa+k_1-1)-\delta^2\left(\kappa-k_1-(k_2-k_1)\left(\lambda\pi+\frac{1-\lambda}{2}\right)\right)}{\kappa-\delta^2\left(\kappa-k_1-(k_2-k_1)\left(\lambda\pi+\frac{1-\lambda}{2}\right)\right)}, \ \textit{and} \\
\widetilde{\theta}_{LO-DO} &\equiv \frac{\delta\frac{1-\lambda}{2}(\kappa+k_1-1)-\delta^2\left(\kappa-k_1-(k_2-k_1)\frac{1-\lambda}{2}\right)}{\kappa-\delta^2\left(\kappa-k_1-(k_2-k_1)\frac{1-\lambda}{2}\right)}.
\end{aligned} \tag{B.3}$$

The cutoffs defined in B.3 are such that

$$\widehat{\theta}_{MO-DO} \leq \overline{\theta}_{MO-DO} < \underline{\theta} \tag{B.4}$$

$$\underline{\theta}_{LO-DO} \leq \min\left\{\widehat{\theta}_{LO-DO}, \widetilde{\theta}_{LO-DO}\right\} \leq \max\left\{\widehat{\theta}_{LO-DO}, \widetilde{\theta}_{LO-DO}\right\} \leq \overline{\theta}_{LO-DO}. \tag{B.5}$$

Lemma 4 *If $k_1 > 1$, then a PBE of the game is as follows:*

- $\mathcal{E}_1^D$*: (BMO, SMO, BLO, SLO) is the optimal strategy profile at $t=1$ if*

$$\kappa_{MO-LO}^{I}\tau \leq \sigma,\ PIN < \psi_{LO-NT}^{U} \text{ and } \theta_1^I \leq \widehat{\theta}_{MO-DO}.$$

The beliefs of an uninformed trader at $t=2$ are: $X^{1,D}=\dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$, $Y^{1,D}=0$ and $Z^{1,D}=z\in[0,1]$. The optimal strategy of an uninformed and an informed trader $t=2$ are *described in Table B.7 and a subset of Table B.5 of Appendix B, respectively.*[26]

- $\mathcal{E}_2^D$: (BMO, SMO, NT, NT) *is the optimal strategy profile at* $t = 1$ *if*

$$\kappa^I_{MO-LO}\tau \leq \sigma, \; PIN \geq \psi^U_{LO-NT}, \; \text{and } \theta^I_1 \leq \overline{\theta}_{MO-DO}.$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{2,D} = \dfrac{\lambda\pi}{1-\lambda+\lambda\pi}$, $Y^{2,D} = p \in [0,1]$ *and* $Z^{2,D} = z \in [0,1]$. *The optimal strategy of an uninformed and an informed trader at* $t = 2$ *are described in Table B.8 and a subset of Table B.5 of Appendix B, respectively.*

- $\mathcal{E}_3^D$: (BLO, SLO, BLO, BLO) *is the optimal strategy profile at* $t = 1$ *if*

$$\begin{aligned}
&\sigma < \kappa^I_{MO-LO}\tau, \; PIN < \psi^U_{LO-NT}, \; \text{and } \theta^I_1 \leq \min\{\underline{\theta}, \widehat{\theta}_{LO-DO}\},\\
&\text{or}\\
&PIN < \psi^U_{LO-NT} \text{ and } \underline{\theta} < \theta^I_1 \leq \min\{\overline{\theta}, \underline{\theta}_{LO-DO}\},\\
&\text{or}\\
&\overline{\theta} < \theta^I_1 \leq \underline{\theta}_{LO-DO}.
\end{aligned}$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{3,D} = 0$, $Y^{3,D} = \pi$ *and* $Z^{3,D} = z \in [0,1]$. *The optimal strategy of an uninformed and an informed trader* $t = 2$ *are described in Table B.9 and a subset of Table B.5 of Appendix B, respectively.*

- $\mathcal{E}_4^D$: (BLO, SLO, NT, NT) *is the optimal strategy profile of a trader at* $t = 1$ *if*

$$\begin{aligned}
&\sigma < \kappa^I_{MO-LO}\tau, \; PIN \geq \psi^U_{LO-NT}, \; \text{and } \theta^I_1 \leq \min\{\underline{\theta}, \overline{\theta}_{LO-DO}\},\\
&\text{or}\\
&PIN \geq \psi^U_{LO-NT} \text{ and } \underline{\theta} < \theta^I_1 \leq \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\}.
\end{aligned}$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{4,D} = 0$, $Y^{4,D} = 1$ *and* $Z^{4,D} = z \in [0,1]$. *The optimal strategy of an uninformed and an informed trader at* $t = 2$ *are described in Table B.10 and a subset of Table B.5 of Appendix B, respectively.*

- $\mathcal{E}_5^D$: (BDO, SDO, BLO, SLO) *is the optimal strategy profile of a trader at* $t = 1$ *if*

$$\begin{aligned}
&PIN < \psi^U_{LO-NT}, \; \theta^I_1 > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\}, \; \text{and } \theta^I_2 \leq \underline{\theta},\\
&\text{or}\\
&PIN < \psi^U_{LO-NT}, \; \widetilde{\theta}_{LO-DO} < \theta^I_1, \; \text{and } \underline{\theta} < \theta^I_2 \leq \overline{\theta},\\
&\text{or}\\
&\widetilde{\theta}_{LO-DO} < \theta^I_1, \; \text{and } \overline{\theta} < \theta^I_2.
\end{aligned}$$

The beliefs of an uninformed trader at $t = 2$ *are:* $X^{5,D} = 0$, $Y^{5,D} = 0$ *and* $Z^{5,D} = 1$. *The*

[26] In the proof of the lemma, we describe for each equilibrium the relevant subset of Table B.5.

optimal strategy of an uninformed and an informed trader $t=2$ are described in Table B.11 and a subset of Table B.5 of Appendix B, respectively.

- $\mathcal{E}_6^D$: *(BDO, SDO, NT, NT) is the optimal strategy profile of a trader at $t=1$ if*

$$PIN \geq \psi^U_{LO-NT},\ \theta^I_1 > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} \text{ and } \theta^I_2 \leq \underline{\theta},$$
$$\text{or}$$
$$PIN \geq \psi^U_{LO-NT},\ \widetilde{\theta}_{LO-DO} < \theta^I_1, \text{ and } \underline{\theta} < \theta^I_2 \leq \overline{\theta}.$$

The beliefs of an uninformed trader at $t=2$ are: $X^{6,D}=0$, $Y^{6,D}=p\in[0,1]$ and $Z^{6,D}=1$. The optimal strategy of an uninformed and an informed trader at $t=2$ are described in Table B.12 and a subset of Table B.5 of Appendix B, respectively.

Remark 3 *Recall that in a Perfect Bayesian Equilibrium beliefs must satisfy Bayes' rule, whenever possible. This occurs along the equilibrium path, not off-the-equilibrium path, where beliefs are indeterminate. This indeterminacy might result in multiplicity of equilibria in sequential games with imperfect information. Note that this may occur in our case when the uninformed trader's beliefs at $t=2$ X, Y, or Z are indeterminate.*

Proof of Lemma 4. Because of the symmetry of the model, without any loss of generality, at $t=1$ we focus on buyers. We present the proof for one of the possible strategy profile at $t=1$ that yields an equilibrium. The proofs of all the other 5 equilibria can be obtained on request from the authors. Note that in all equilibria the optimal responses of informed traders at $t=2$ are given in Table B.5. However, in some equilibria not all the 6 cases $I_1 - I_6$ are possible and also not all of the 5 states of the book are possible. As a result only a subset of Table B.5 will apply.

$\mathcal{E}_1^D$**:** (BMO, SMO, BLO, SLO)

First step. In this case $\Omega_0=0$, $\Omega_1=1$, $\Omega_2=0$, $\Omega_3=0$, $\Gamma_0=0$, $\Gamma_1=0$, $\Gamma_2=1$, and $\Gamma_3=0$. Moreover, $\theta^I_2=\theta^I_1$ and $\theta^U_2=\theta^U_1$.

Second step. Using Bayes's rule

$$\begin{aligned} X^{1,D} &= \frac{\lambda\pi}{1-\lambda+\lambda\pi},\ Y^{1,D}=0,\ Z^{1,D}=z\in[0,1]\,, \\ p^{UB,\mathcal{B}_1=\varnothing}_{BLO,2} &= p^{US,\mathcal{B}_1=\varnothing}_{SLO,2}\in[0,1]\,, \text{ and } p^{IH,\mathcal{B}_1=\varnothing}_{BLO,2}=p^{IL,\mathcal{B}_1=\varnothing}_{SLO,2}\in[0,1]\,. \end{aligned}$$

Third step. Using step 2 and taking into account that $p^{UB}_{BLO,2}(\mathcal{B}_1=\varnothing)=p^{US}_{SLO,2}(\mathcal{B}_1=\varnothing)\in[0,1]$, at $t=2$ the expected profits of uninformed buyers are as given by Table B.2. Using the symmetry of buyers and sellers, we obtain that the optimal strategy for the uninformed are:

Optimal Strategies of Uninformed Traders at $t=2$		
State of the Book	**UB**	**US**
(A_1^1, B_1^1)	$\begin{cases} NT & \text{if } p_{BLO,2}^{UB,\mathcal{B}_1=\varnothing}=0 \text{ or } Z^{1,D} \geq \frac{k_1-1}{\kappa} \\ BLO & \text{if } p_{BLO,2}^{UB,\mathcal{B}_1=\varnothing}>0 \text{ and } Z^{1,D} < \frac{k_1-1}{\kappa} \end{cases}$	$\begin{cases} NT & \text{if } p_{BLO,2}^{UB,\mathcal{B}_1=\varnothing}=0 \text{ or } Z^{1,D} \geq \frac{k_1-1}{\kappa} \\ SLO & \text{if } p_{BLO,2}^{UB,\mathcal{B}_1=\varnothing}>0 \text{ and } Z^{1,D} < \frac{k_1-1}{\kappa} \end{cases}$
(A_1^2, B_1^1)	$\begin{cases} NT & \text{if } X^{1,D}\kappa \leq \frac{k_2-k_1}{2} \\ BDO & \text{if } \frac{k_2-k_1}{2} < X^{1,D}\kappa \leq k_2 \\ BDO & \text{if } k_2 < X^{1,D}\kappa \text{ and } \theta_2^U > \theta_{X^{1,D}} \\ BMO & \text{if } k_2 < X^{1,D}\kappa \text{ and } \theta_2^U \leq \theta_{X^{1,D}} \end{cases}$	$\begin{cases} SDO & \text{if } X^{1,D}\kappa < \frac{k_2-k_1}{2} \\ NT & \text{if } \frac{k_2-k_1}{2} \leq X^{1,D}\kappa \end{cases}$
$(A_1^1, B_1^1+\tau)$	NT	SDO
(A_1^1, B_1^2)	$\begin{cases} BDO & \text{if } X^{1,D}\kappa < \frac{k_2-k_1}{2} \\ NT & \text{if } \frac{k_2-k_1}{2} \leq X^{1,D}\kappa \end{cases}$	$\begin{cases} NT & \text{if } X^{1,D}\kappa \leq \frac{k_2-k_1}{2} \\ SDO & \text{if } \frac{k_2-k_1}{2} < X^{1,D}\kappa \leq k_2 \\ SDO & \text{if } k_2 < X^{1,D}\kappa \text{ and } \theta_2^U > \theta_{X^{1,D}} \\ SMO & \text{if } k_2 < X^{1,D}\kappa \text{ and } \theta_2^U \leq \theta_{X^{1,D}} \end{cases}$
$(A_1^1-\tau, B_1^1)$	BDO	NT

Table B.6: Optimal strategies of uninformed traders at $t=2$ when the strategy profile at $t=1$ is (BMO, SMO, BLO, SLO).

Concerning the informed buyers their expected profits are as given by Table B.1 and the optimal strategy for an informed trader at $t=2$ is given by Table B.5.

Fourth step. Given the optimal response of traders at $t=2$, find the optimal action for the traders at $t=1$ in each of the 6 cases. However, given the nature of this particular equilibrium, we can group cases and analyze them in the following way:

Case $I_1+I_2+I_3 : \theta_2^I \leq \dfrac{\kappa-k_1}{\kappa}$

- *Informed traders*

  As $\theta_2^I \leq \dfrac{\kappa-k_1}{\kappa}$, informed traders at $t=1$ have no incentives to deviate from the prescribed strategy profile whenever

$$\begin{aligned} \kappa - k_1 &\geq \frac{1-\lambda}{2}\delta\left(\kappa+k_1-1\right) \text{ and} \\ \kappa - k_1 &\geq \theta_1^I\kappa + (1-\theta_1^I)\delta^2\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2}I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2-k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right). \end{aligned}$$

- *Uninformed traders*

  As $\theta_2^I \leq \dfrac{\kappa - k_1}{\kappa} \leq \dfrac{\kappa - k_1 + 1}{\kappa + \frac{1}{2}}$, uninformed traders at $t = 1$ have no incentives to deviate from the prescribed strategy profile whenever

$$(\lambda\pi + 1 - \lambda)(k_1 - 1) - \lambda\pi\kappa > 0.$$

Case $I_4 + I_5 + I_6 : \dfrac{\kappa - k_1}{\kappa} < \theta_2^I$

- *Informed traders*

Consider an informed buyer at $t = 1$. If he chooses a BMO, then he obtains

$$\mathbb{E}\left(\Pi_{BMO,1}^{IH}\right) = (\kappa - k_1)\,\tau.$$

If instead he deviates towards a BDO, he will obtain

$$\begin{aligned} \mathbb{E}\left(\Pi_{BDO,1}^{IH}\right) &= \theta_1^I \kappa\tau + (1 - \theta_1^I)\delta^2 \left[\lambda \frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} + (\kappa - k_1) \right. \\ &\left. -(k_2 - k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right]\tau. \end{aligned}$$

Combining the previous expression and the fact that $\dfrac{\kappa - k_1}{\kappa} < \theta_2^I = \theta_1^I$, it follows that

$$\mathbb{E}\left(\Pi_{BDO,1}^{IH}\right) > \mathbb{E}\left(\Pi_{BMO,1}^{IH}\right)$$

is always satisfied and, hence, we conclude that in this case there is no equilibrium in which (BMO, SMO, BLO, SLO) is the strategy profile chosen at $t = 1$.

Fifth step. Based on the above, nobody at $t = 1$ has unilateral incentives to deviate whenever

$$\theta_1^I \leq \frac{\kappa - k_1}{\kappa},$$
$$(\lambda\pi + 1 - \lambda)(k_1 - 1) - \lambda\pi\kappa > 0,$$
$$\kappa - k_1 \geq \frac{1-\lambda}{2}\delta(\kappa + k_1 - 1) \text{ and}$$
$$\kappa - k_1 \geq \theta_1^I\kappa + (1-\theta_1^I)\delta^2\left(\kappa - k_1 + \lambda\frac{(1-\pi)}{2} I_{SLO,2}^{US,\mathcal{B}_1=\varnothing} - (k_2 - k_1)\left(\lambda\pi I_{BMO,2}^{IH,\mathcal{B}_1=\varnothing} + \frac{1-\lambda}{2}\right)\right).$$

Using the definition $PIN = \pi\lambda$, these conditions are equivalent to

$$\theta_1^I \leq \underline{\theta}$$
$$\kappa_{MO-LO}^I \tau \leq \sigma \text{ and } PIN < \psi_{LO-NT}^U$$
$$\theta_1^I \leq \widehat{\theta}_{MO-DO},$$

where $\underline{\theta}$ and $\widehat{\theta}_{MO-DO}$ are defined in (B.3). Notice also that from (B.4) we have that $\widehat{\theta}_{MO-DO} < \underline{\theta}$ and, therefore, we can simplify further to

$$\begin{gathered}\kappa^I_{MO-LO}\tau \leq \sigma \text{ and } PIN < \psi^U_{LO-NT} \\ \theta^I_1 \leq \widehat{\theta}_{MO-DO}.\end{gathered} \tag{B.6}$$

Finally, in the following tables we include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BMO, SMO, BLO, SLO) is the strategy profile chosen at $t = 1$ and the fact that in this case $\theta^I_2 = \theta^I_1$.

Concerning uninformed traders notice that the condition $(\lambda\pi + 1 - \lambda)(k_1 - 1) - \lambda\pi\kappa > 0$ implies that $X^{1,D}\kappa < k_1 - 1 < k_2$. Hence, the optimal choice of uninformed traders at $t = 2$ is as follows:

Condition	**Optimal Choice of Uninformed Traders at t = 2**		
	State of the Book	**UB**	**US**
Case $U_1^{\mathcal{E}_1^D}$ $k_1 - 1 \leq \frac{k_2-k_1}{2}$ or $k_1 - 1 > \frac{k_2-k_1}{2}$ and $X^{1,D}\kappa < \frac{k_2-k_1}{2}$	(A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT NT BDO BDO	SDO SDO NT NT
Case $U_2^{\mathcal{E}_1^D}$ $k_1 - 1 > \frac{k_2-k_1}{2}$ and $X^{1,D}\kappa = \frac{k_2-k_1}{2}$	(A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	NT NT NT BDO	NT SDO NT NT
Case $U_3^{\mathcal{E}_1^D}$ $k_1 - 1 > \frac{k_2-k_1}{2}$ and $\frac{k_2-k_1}{2} < X^{1,D}\kappa < k_1 - 1$	(A_1^2, B_1^1) $(A_1^1, B_1^1 + \tau)$ (A_1^1, B_1^2) $(A_1^1 - \tau, B_1^1)$	BDO NT NT BDO	NT SDO SDO NT

Table B.7: Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, BLO, SLO)

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table B.5 only the cases I_1, I_2 and I_3 and the following possible prices (A_1^2, B_1^1), $(A_1^1, B_1^1 + \tau)$, (A_1^1, B_1^2), $(A_1^1 - \tau, B_1^1)$.

$\mathcal{E}_2^D$**:** (BMO, SMO, NT, NT)

Following the same procedure we obtain that in this case nobody at $t = 1$ has unilateral incentives to deviate whenever

$$\kappa^I_{MO-LO}\tau \leq \sigma,\ PIN \geq \psi^U_{LO-NT}, \text{ and } \theta^I_1 \leq \overline{\theta}_{MO-DO}. \tag{B.7}$$

Finally, in the following tables we include the decisions that are in the equilibrium path taking into account the conditions that must be satisfied if (BMO, SMO, NT, NT) is the strategy profile chosen at $t = 1$ and that in this case $\theta_2^I = \theta_1^I$.

In relation to uninformed traders, and taking into account that a necessary condition for this equilibrium tells us that $k_1 - 1 \leq X^{2,D}\kappa$, the following cases can be distinguished:

Condition	**Optimal Choice of Uninformed Traders at t = 2**		
	State of the Book	**UB**	**US**
Case $U_1^{\mathcal{E}_2^D}$ $k_1 - 1 \leq X^{2,D}\kappa < \frac{k_2-k_1}{2}$	(A_1^1, B_1^1) (A_1^2, B_1^1) (A_1^1, B_1^2)	NT NT BDO	NT SDO NT
Case $U_2^{\mathcal{E}_2^D}$ $k_1 - 1 < X^{2,D}\kappa = \frac{k_2-k_1}{2}$	(A_1^1, B_1^1) (A_1^2, B_1^1) (A_1^1, B_1^2)	NT NT NT	NT NT NT
Case $U_3^{\mathcal{E}_2^D}$ $\max\left\{k_1 - 1, \frac{k_2-k_1}{2}\right\} < X^{2,D}\kappa \leq k_2$ or $k_2 < X^{2,D}\kappa$ and $\theta_2^U > \theta_{X^{2,D}}$	(A_1^1, B_1^1) (A_1^2, B_1^1) (A_1^1, B_1^2)	NT BDO NT	NT NT SDO
Case $U_4^{\mathcal{E}_2^D}$ $k_2 < X^{2,D}\kappa$ and $\theta_2^U \leq \theta_{X^{2,D}}$	(A_1^1, B_1^1) (A_1^2, B_1^1) (A_1^1, B_1^2)	NT BMO NT	NT NT SMO

Table B.8: Optimal choice of uniformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BMO, SMO, NT, NT)

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table B.5 only the cases I_1, I_2 and I_3 and the following possible prices $(A_1^1, B_1^1), (A_1^2, B_1^1), (A_1^1, B_1^2)$.

$\mathcal{E}_3^D$**:** (BLO, SLO, BLO, BLO)

Following the same procedure we obtain that the conditions under which nobody is willing to deviate at $t = 1$ are:

$$\begin{aligned} & \sigma < \kappa_{MO-LO}^I \tau, \; PIN < \psi_{LO-NT}^U, \text{ and } \theta_1^I \leq \min\{\underline{\theta}, \widehat{\theta}_{LO-DO}\}, \\ \text{or} \quad & PIN < \psi_{LO-NT}^U \text{ and } \underline{\theta} < \theta_1^I \leq \min\{\overline{\theta}, \underline{\theta}_{LO-DO}\}, \\ \text{or} \quad & \overline{\theta} < \theta_1^I \leq \underline{\theta}_{LO-DO}. \end{aligned} \tag{B.8}$$

Finally, in the following tables we include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BLO, SLO, BLO, SLO) is the strategy profile chosen at $t = 1$ and the fact that in this case $\theta_2^I = \theta_1^I$.

Concerning uninformed traders, it follows that their optimal choices at $t = 2$ are

Condition	Optimal Choice of Uninformed Traders at t = 2		
	State of the Book	**UB**	**US**
Case $U_1^{\mathcal{E}_3^D}$ $Y^{3,D}\kappa \leq \frac{1}{2}$	(A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT NT BDO BDO	SDO SDO NT NT
Case $U_2^{\mathcal{E}_3^D}$ $\frac{1}{2} < Y^{3,D}\kappa \leq k_1$ or $Y^{3,D}\kappa > k_1$and $\theta_2^U > \theta_{Y^{3,D}}$	(A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT BDO BDO NT	SDO NT NT SDO
Case $U_3^{\mathcal{E}_3^D}$ $Y^{3,D}\kappa > k_1$ and $\theta_2^U \leq \theta_{Y^{3,D}}$	(A_1^2, B_1^1) $(A_1^1, B_1^1+\tau)$ (A_1^1, B_1^2) $(A_1^1-\tau, B_1^1)$	NT BMO BDO NT	SDO NT NT SMO

Table B.9: Optimal choice of uninformed traders when the strategy profile at $t = 1$ is (BLO, SLO, BLO, SLO).

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table B.5 all the cases I_1- I_6 and the following possible prices (A_1^2, B_1^1), $(A_1^1, B_1^1+\tau)$, (A_1^1, B_1^2), $(A_1^1-\tau, B_1^1)$.

$\mathcal{E}_4^D$: (BLO, SLO, NT, NT)

Following the same procedure we obtain that the conditions under which nobody is willing to deviate at $t = 1$ are:

$$\begin{aligned} & \sigma < \kappa_{MO-LO}^I\tau,\ PIN \geq \psi_{LO-NT}^U,\ \text{and } \theta_1^I \leq \min\{\underline{\theta}, \overline{\theta}_{LO-DO}\}, \\ \text{or} \quad & PIN \geq \psi_{LO-NT}^U \text{ and } \underline{\theta} < \theta_1^I \leq \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\}. \end{aligned} \tag{B.9}$$

Finally, in the following tables we include the moves that are in the equilibrium path taking into account the conditions that must be satisfied if (BLO, SLO, NT, NT) is the strategy profile chosen at $t = 1$ and the fact that in this case $\theta_2^I = \theta_1^I$ and $\theta_2^U = \theta_1^U$. Concerning the uninformed traders, we have

Condition	Optimal Choice of Uninformed Traders at t = 2		
	State of the Book	**UB**	**US**
Case $U_1^{\mathcal{E}_4^D}$ $\theta_2^U > \theta_{Y4,D}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1 + \tau)$	BDO	NT
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1 - \tau, B_1^1)$	NT	SDO
Case $U_2^{\mathcal{E}_4^D}$ $\theta_2^U \leq \theta_{Y4,D}$	(A_1^1, B_1^1)	NT	NT
	(A_1^2, B_1^1)	NT	SDO
	$(A_1^1, B_1^1 + \tau)$	BMO	NT
	(A_1^1, B_1^2)	BDO	NT
	$(A_1^1 - \tau, B_1^1)$	NT	SMO

Table B.10: Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BLO, SLO, NT, NT).

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table B.5 all the cases $I_1 -$ I_6 and for the entire book but with $BX = BMO$, $SX = SMO$, $BY = BDO$, $SX = SDO$.

$\mathcal{E}_5^D$**:** (BDO, SDO, BLO, SLO)
Following the same procedure we obtain that in this case nobody at $t = 1$ has unilateral incentives to deviate from (BDO, SDO, BLO, SLO) whenever

$$\begin{aligned} & PIN < \psi^U_{LO-NT},\ \theta_1^I > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\}, \text{ and } \theta_2^I \leq \underline{\theta}, \\ \text{or} \quad & PIN < \psi^U_{LO-NT},\ \widetilde{\theta}_{LO-DO} < \theta_1^I, \text{ and } \underline{\theta} < \theta_2^I \leq \overline{\theta}, \\ \text{or} \quad & \widetilde{\theta}_{LO-DO} < \theta_1^I, \text{ and } \overline{\theta} < \theta_2^I. \end{aligned} \tag{B.10}$$

Notice that in this equilibrium we always have $\theta_2^I < \theta_1^I$.
The optimal responses of uninformed traders are:

State of the book	UB	US
(A_1^1, B_1^1)	NT	NT
(A_1^2, B_1^1)	NT	SDO
$(A_1^1, B_1^1 + \tau)$	NT	SDO
(A_1^1, B_1^2)	BDO	NT
$(A_1^1 - \tau, B_1^1)$	BDO	NT

Table B.11: Optimal responses of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BDO, SDO, BLO, SLO).

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table

B.5 all the cases $I_1 -$ I_6 and for the entire book but with $BX = BMO$, $SX = SMO$, $BY = BDO$, $SX = SDO$.

$\mathcal{E}_6^D$**:** (BDO, SDO, NT, NT)

Following the same procedure we obtain that in this case nobody at $t = 1$ has unilateral incentives to deviate from (BDO, SDO, NT, NT) whenever

$$\begin{aligned} & PIN \geq \psi^U_{LO-NT},\ \theta^I_1 > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\}, \text{ and } \theta^I_2 \leq \underline{\theta}, \\ \text{or} \quad & PIN \geq \psi^U_{LO-NT},\ \widetilde{\theta}_{LO-DO} < \theta^I_1, \text{ and } \underline{\theta} < \theta^I_2 \leq \overline{\theta}. \end{aligned} \tag{B.11}$$

Notice that in this equilibrium we also have that $\theta^I_2 \leq \theta^I_1$.

The optimal responses of uninformed traders are:

Optimal Choice of Uninformed Traders at $t = 2$		
State of the Book	**UB**	**US**
(A^1_1, B^1_1)	NT	NT
(A^2_1, B^1_1)	NT	SDO
(A^1_1, B^2_1)	BDO	NT

Table B.12: Optimal choice of uninformed traders at $t = 2$ when the strategy profile at $t = 1$ is (BDO, SDO, NT, NT).

In relation to informed traders the optimal choice at $t = 2$ can be obtained by selecting in Table B.5 all the cases $I_1 - I_6$ and the following possible prices (A^1_1, B^1_1), (A^2_1, B^1_1), (A^1_1, B^2_1). ■

Proof of Proposition 2. We consider the same four possible cases, depending on the initial conditions in the market before the introduction of the dark pool.

Case 1: $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN < \psi^U_{LO-NT}$

In this case, we start with a market in which the equilibrium is $\mathcal{E}_3^{ND}$, where conditions $(A.8)$ and $(A.9)$ are satisfied. In this case, when we add the dark pool out of the 6 equilibria, there are only two possible equilibria that satisfy these conditions: $\mathcal{E}_3^D$ and $\mathcal{E}_5^D$. From Lemma 4 we can see that $\mathcal{E}_3^D$ is an equilibrium if conditions $(B.8)$ are satisfied. This can be rewritten as

$$\begin{gathered} \theta^I_1 \leq \min\{\underline{\theta}, \widehat{\theta}_{LO-DO}\} \\ \text{or} \\ \underline{\theta} < \theta^I_1 \leq \underline{\theta}_{LO-DO}. \end{gathered}$$

Using (B.5), we know that $\underline{\theta}_{LO-DO} \leq \widehat{\theta}_{LO-DO}$. Then, we consider the following cases: I) $\underline{\theta} < \underline{\theta}_{LO-DO}$, II) $\underline{\theta}_{LO-DO} \leq \underline{\theta} < \widehat{\theta}_{LO-DO}$, and III) $\widehat{\theta}_{LO-DO} \leq \underline{\theta}$.

Case I: $\underline{\theta} < \underline{\theta}_{LO-DO}$. In this case, $\underline{\theta} < \widehat{\theta}_{LO-DO}$. Hence, the conditions that guarantee that $\mathcal{E}_3^D$

is an equilibrium can be rewritten as

$$\begin{aligned}&\theta_1^I \leq \underline{\theta}\\ &\text{or}\\ &\underline{\theta} < \theta_1^I \leq \underline{\theta}_{LO-DO},\end{aligned}$$

which can be further simplified as

$$\theta_1^I \leq \underline{\theta}_{LO-DO}.$$

Case II: $\underline{\theta}_{LO-DO} \leq \underline{\theta} < \widehat{\theta}_{LO-DO}$. In this case, the conditions that guarantee that $\mathcal{E}_3^D$ is an equilibrium can be rewritten as

$$\theta_1^I \leq \underline{\theta}_{LO-DO}.$$

Case III: $\widehat{\theta}_{LO-DO} \leq \underline{\theta}$. In this case, the conditions that guarantee that $\mathcal{E}_3^D$ is an equilibrium can be rewritten as

$$\theta_1^I \leq \widehat{\theta}_{LO-DO}.$$

Consequently, we conclude that $\mathcal{E}_3^D$ is an equilibrium whenever

$$\begin{aligned}&\theta_1^I \leq \underline{\theta}_{LO-DO} \quad \text{if } \underline{\theta} < \widehat{\theta}_{LO-DO},\\ &\text{or}\\ &\theta_1^I \leq \widehat{\theta}_{LO-DO} \quad \text{if } \widehat{\theta}_{LO-DO} \leq \underline{\theta}.\end{aligned}$$

On the other hand, $\mathcal{E}_5^D$ is an equilibrium if conditions $(B.10)$ are satisfied, and in this case they can be rewritten as

$$\begin{aligned}&\theta_1^I > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} \text{ and } \theta_2^I \leq \underline{\theta},\\ &\text{or}\\ &\widetilde{\theta}_{LO-DO} < \theta_1^I \text{ and } \underline{\theta} < \theta_2^I.\end{aligned}$$

Notice that when $\sigma < \kappa_{MO-LO}^I \tau$, the informed traders prefer LO to MO and, therefore, $\overline{\theta}_{MO-DO} < \overline{\theta}_{LO-DO}$. Hence, $\mathcal{E}_5^D$ the equilibrium if

$$\begin{aligned}&\theta_1^I > \overline{\theta}_{LO-DO} \text{ and } \theta_2^I \leq \underline{\theta},\\ &\text{or}\\ &\widetilde{\theta}_{LO-DO} < \theta_1^I \text{ and } \underline{\theta} < \theta_2^I.\end{aligned}$$

As a result, when $\sigma < \kappa_{MO-LO}^I \tau$ and $PIN < \psi_{LO-NT}^U$, the optimal strategy profiles at $t = 1$ are

$$\begin{cases} (BLO, SLO, BLO, SLO), & \text{if} \quad \theta_1^I \leq \theta_{LO-LO}^1, \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta_1^I > \theta_{DO-LO}^1, \end{cases}$$

where

$$\theta^1_{LO-LO} = \begin{cases} \widehat{\theta}_{LO-DO} & \text{if} \quad \widehat{\theta}_{LO-DO} < \underline{\theta} \\ \underline{\theta}_{LO-DO} & \text{otherwise} \end{cases}, \text{ and} \tag{B.12}$$

$$\theta^1_{DO-LO} = \begin{cases} \overline{\theta}_{LO-DO} & \text{if} \quad \theta^I_2 \le \underline{\theta} \\ \widetilde{\theta}_{LO-DO} & \text{otherwise.} \end{cases}$$

Case 2: $\sigma < \kappa^I_{MO-LO}\tau$ and $PIN \ge \psi^U_{LO-NT}$

In this case, when we start with a market in which the equilibrium is $\mathcal{E}^{ND}_4$, where conditions $(A.10)$ and $(A.11)$ are satisfied. In this case, when we add the dark pool out of the 6 equilibria there are only three possible equilibria that satisfy these conditions: $\mathcal{E}^D_4$, $\mathcal{E}^D_5$, and $\mathcal{E}^D_6$. From Lemma 4 we can see that $\mathcal{E}^D_4$ is an equilibrium if conditions $(B.9)$ are satisfied, and in this case they can be rewritten as

$$\begin{aligned} &\theta^I_1 \le \min\{\underline{\theta}, \overline{\theta}_{LO-DO}\} \\ &\text{or } \underline{\theta} < \theta^I_1 \le \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\}. \end{aligned}$$

Consider the following cases: I) $\underline{\theta} < \overline{\theta}_{LO-DO}$ and II) $\overline{\theta}_{LO-DO} \le \underline{\theta}$.

Case I: $\underline{\theta} < \overline{\theta}_{LO-DO}$. In this case, the conditions that guarantee that $\mathcal{E}^D_4$ is an equilibrium can be rewritten as

$$\begin{aligned} &\theta^I_1 \le \underline{\theta} \\ &\text{or } \underline{\theta} < \theta^I_1 \le \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\}, \end{aligned}$$

i.e.,

$$\theta^I_1 \le \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\}.$$

Case II: $\overline{\theta}_{LO-DO} \le \underline{\theta}$. In this case, $\widetilde{\theta}_{LO-DO} < \underline{\theta}$. Thus, the conditions that guarantee that $\mathcal{E}^D_4$ is an equilibrium can be rewritten as

$$\theta^I_1 \le \overline{\theta}_{LO-DO}.$$

On the other hand, $\mathcal{E}^D_5$ is an equilibrium if conditions $(B.10)$ are satisfied, and in this case they can be rewritten as

$$\widetilde{\theta}_{LO-DO} < \theta^I_1 \text{ and } \overline{\theta} < \theta^I_2.$$

Finally, $\mathcal{E}^D_6$ is an equilibrium if

$$\begin{aligned} &\theta^I_1 > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} \text{ and } \theta^I_2 \le \underline{\theta}, \\ &\text{or} \\ &\widetilde{\theta}_{LO-DO} < \theta^I_1, \text{ and } \underline{\theta} < \theta^I_2 \le \overline{\theta}. \end{aligned}$$

Given that $\sigma < \kappa^I_{MO-LO}\tau$, we know that the informed prefer LO to MO. Hence, $\overline{\theta}_{LO-DO} > \overline{\theta}_{MO-DO}$, which implies $\max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} = \overline{\theta}_{LO-DO}$. Thus, the previous conditions can be rewritten as

$$\theta_1^I > \overline{\theta}_{LO-DO} \text{ and } \theta_2^I \leq \underline{\theta},$$
$$\text{or}$$
$$\widetilde{\theta}_{LO-DO} < \theta_1^I, \text{ and } \underline{\theta} < \theta_2^I \leq \overline{\theta}.$$

As a result, the optimal strategy profiles of a trader at $t = 1$ are

$$\begin{cases} (BLO, SLO, NT, NT) & \text{if} \quad \theta_1^I \leq \theta_{LO-NT}^{22} \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta_{DO-NT}^{22} < \theta_1^I \leq \theta_{DO-LO}^{22} \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta_1^I > \theta_{DO-LO}^{22}, \end{cases}$$

where

$$\theta_{LO-NT}^{22} = \begin{cases} \min\{\overline{\theta}, \widetilde{\theta}_{LO-DO}\} & \text{if} \quad \underline{\theta} < \overline{\theta}_{LO-DO} \\ \overline{\theta}_{LO-DO} & \text{otherwise,} \end{cases} \tag{B.13}$$

$$\theta_{DO-NT}^{22} = \begin{cases} \overline{\theta}_{LO-DO} & \text{if} \quad \theta_2^I \leq \underline{\theta} \\ \widetilde{\theta}_{LO-DO} & \text{if} \quad \underline{\theta} < \theta_2^I \leq \overline{\theta} \quad \text{and} \\ 1 & \text{if} \quad \overline{\theta} < \theta_2^I, \end{cases}$$

$$\theta_{DO-LO}^{22} = \begin{cases} 1 & \text{if} \quad \theta_2^I \leq \overline{\theta} \\ \widetilde{\theta}_{LO-DO} & \text{if} \quad \text{otherwise.} \end{cases}$$

Case 3: $\kappa_{MO-LO}^I \tau \leq \sigma$ and $PIN < \psi_{LO-NT}^U$

In this case we start with a market in which the equilibrium is $\mathcal{E}_1^{ND}$, where conditions $(A.3)$ and $(A.4)$ are satisfied. In this case, when we add the dark pool out of the 6 equilibria there are only two possible equilibria that satisfy these conditions: $\mathcal{E}_1^D$ and $\mathcal{E}_5^D$. From Lemma 4 we can see that $\mathcal{E}_1^D$ is an equilibrium if conditions $(B.6)$ are satisfied. Similarly, $\mathcal{E}_5^D$ is an equilibrium if conditions $(B.10)$ are satisfied, and in this case they can be rewritten as

$$\theta_1^I > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} \text{ and } \theta_2^I \leq \underline{\theta},$$
$$\text{or}$$
$$\widetilde{\theta}_{LO-DO} < \theta_1^I \text{ and } \underline{\theta} < \theta_2^I.$$

Given that $\kappa_{MO-LO}^I \tau \leq \sigma$, we have that informed traders prefer MO to LO. Consequently, $\overline{\theta}_{MO-DO} \geq \overline{\theta}_{LO-DO}$ and therefore we have that $\mathcal{E}_5^D$ is an equilibrium if

$$\theta_1^I > \overline{\theta}_{MO-DO} \text{ and } \theta_2^I \leq \underline{\theta},$$
$$\text{or}$$
$$\widetilde{\theta}_{LO-DO} < \theta_1^I \text{ and } \underline{\theta} < \theta_2^I.$$

As a result in this case the the optimal strategy profiles of a trader at $t = 1$ are :

$$\begin{cases} (BMO, SMO, BLO, BLO) & \text{if} \quad \theta_1^I \leq \widehat{\theta}_{MO-DO} \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta_1^I > \theta_{DO-LO}^{21}, \end{cases}$$

where

$$\theta_{DO-LO}^{21} = \begin{cases} \overline{\theta}_{MO-DO} & \text{if} \quad \theta_2^I \leq \underline{\theta} \\ \widetilde{\theta}_{LO-DO} & \text{otherwise.} \end{cases} \tag{B.14}$$

Case 4: $\kappa_{MO-LO}^I \tau \leq \sigma$ and $PIN \geq \psi_{LO-NT}^U$

In this case we start with a market in which the equilibrium is $\mathcal{E}_2^{ND}$, where conditions $(A.6)$ and $(A.7)$ are satisfied. In this case, when we add the dark pool out of the 6 equilibria there are only three possible equilibria that satisfy these conditions: $\mathcal{E}_2^D$, $\mathcal{E}_5^D$, and $\mathcal{E}_6^D$. From Lemma 4 we can see that $\mathcal{E}_2^{ND}$ is an equilibrium if conditions $(B.7)$ are satisfied. Similarly, $\mathcal{E}_5^D$ is an equilibrium if conditions $(B.10)$ are satisfied, and in this case they can be rewritten as

$$\widetilde{\theta}_{LO-DO} < \theta_1^I \text{and } \overline{\theta} < \theta_2^I.$$

Finally, $\mathcal{E}_6^D$ is an equilibrium if conditions $(B.11)$ are satisfied and in this case they can be rewritten as

$$\begin{aligned} &\theta_1^I > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} \text{ and } \theta_2^I \leq \underline{\theta}, \\ &\text{or} \\ &\widetilde{\theta}_{LO-DO} < \theta_1^I, \text{ and } \ \underline{\theta} < \theta_2^I \leq \overline{\theta}. \end{aligned}$$

Given that $\kappa_{MO-LO}^I \tau \leq \sigma$, it follows that informed prefer MO to LO and, therefore, $\overline{\theta}_{LO-DO} \leq \overline{\theta}_{MO-DO}$, which implies $\max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} = \overline{\theta}_{MO-DO}$. Hence, the previous conditions can be rewritten as

$$\begin{aligned} &\theta_1^I > \overline{\theta}_{MO-DO} \text{ and } \theta_2^I \leq \underline{\theta}, \\ &\text{or} \\ &\widetilde{\theta}_{LO-DO} < \theta_1^I, \text{ and } \ \underline{\theta} < \theta_2^I \leq \overline{\theta}. \end{aligned}$$

As a result, the optimal strategy profiles of a trader at $t = 1$ are

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta_1^I \leq \overline{\theta}_{MO-DO} \\ \\ (BDO, SDO, NT, NT) & \text{if} \quad \theta_{DO-NT}^3 < \theta_1^I \leq \theta_{DO-LO}^3 \\ \\ (BDO, SDO, BLO, BLO) & \text{if} \quad \theta_1^I > \theta_{DO-LO}^3, \end{cases}$$

where

$$\theta^3_{DO-NT} = \begin{cases} \overline{\theta}_{MO-DO} & \text{if} \quad \theta^I_2 \leq \underline{\theta} \\ \widetilde{\theta}_{LO-DO} & \quad \underline{\theta} < \theta^I_2 \leq \overline{\theta} \\ 1 & \quad \overline{\theta} < \theta^I_2. \end{cases} \tag{B.15}$$

$$\theta^3_{DO-LO} = \begin{cases} 1 & \text{if} \quad \theta^I_2 \leq \overline{\theta} \\ \widetilde{\theta}_{LO-DO} & \text{if} \quad \overline{\theta} < \theta^I_2. \end{cases}$$

■

Proof of Proposition 3. Let us denote by $I^{ND,i}_t$ ($I^{D,i}_t$) the price informativeness in trading period t corresponding to the equilibrium $\mathcal{E}^{ND}_i$ ($\mathcal{E}^D_i$). Note that

$$\begin{aligned} I^{ND,i}_1 &= \mathbb{E}\left(Var\left(V\right) - Var\left(V|\left(A^{ND,i}_2, B^{ND,i}_2\right)\right)\right) \text{ and} \\ I^{ND,i}_2 &= \mathbb{E}\left(Var\left(V\right) - Var\left(V|\left(A^{ND,i}_2, B^{ND,i}_2\right)\right), \left(A^{ND,i}_3, B^{ND,i}_3\right)\right), \end{aligned}$$

where $\left(A^{ND,i}_2, B^{ND,i}_2\right)$ and $\left(A^{ND,i}_3, B^{ND,i}_3\right)$ represent the best prices at the end of period 1 and at the end of period 2, respectively, corresponding to the equilibrium $\mathcal{E}^{ND}_i$. $I^{D,i}_1$ and $I^{D,i}_2$ can be defined analogously.

We calculate the price informativeness in the first period, and find that

$$\begin{aligned} I^{ND,1}_1 &= \frac{\lambda^2\pi^2}{(\lambda\pi + 1 - \lambda)}\kappa^2\tau^2, \\ I^{ND,2}_1 &= \frac{\lambda^2\pi^2}{\lambda\pi + 1 - \lambda}\kappa^2\tau^2, \\ I^{ND,3}_1 &= \lambda\pi^2\kappa^2\tau^2, \\ I^{ND,4}_1 &= \lambda\pi\kappa^2\tau^2, \\ I^{D,j}_1 &= I^{ND,j}_1, \ j = 1, ..., 4, \text{ and} \\ I^{D,5}_1 &= I^{D,6}_1 = 0. \end{aligned}$$

Hence, the effect of the introduction of a dark pool on the price informativeness in trading period 1 can be illustrated by the following table:

Informed traders	$\mathcal{E}^D_1$	$\mathcal{E}^D_2$	$\mathcal{E}^D_3$	$\mathcal{E}^D_4$	$\mathcal{E}^D_5$	$\mathcal{E}^D_6$
$\mathcal{E}^{ND}_1$	$=$				$>$	
$\mathcal{E}^{ND}_2$		$=$			$>$	$>$
$\mathcal{E}^{ND}_3$			$=$		$>$	
$\mathcal{E}^{ND}_4$				$=$	$>$	$>$

Next, we consider the second trading period. In order to show that the results of the comparison on the price informativeness at $t = 2$ are ambiguous, we contrast $\mathcal{E}^{ND}_2$ and $\mathcal{E}^D_2$. After tedious computations we obtain that

$$I_2^{ND,2} = \begin{cases} \frac{\lambda^2\pi^2}{2}\frac{2\lambda^2\pi^2(1-\lambda\pi)+\left(3\lambda^2-4\lambda+5\right)\lambda\pi+(1-\lambda)\left(\lambda^2-\lambda+4\right)}{(\lambda^2\pi^2+\lambda\pi+1-\lambda)(\lambda\pi+1-\lambda)}\kappa^2\tau^2 & \text{if} \quad X_2^{ND} > \frac{k_2}{\kappa} \\ 2\lambda^2\pi^2\frac{-\lambda^3\pi^3+(\lambda+1)\lambda^2\pi^2+2(1-\lambda)\lambda\pi+(\lambda-1)^2}{\left((\lambda\pi+1-\lambda)^2+\lambda^2\pi^2\right)(\lambda\pi+1-\lambda)}\kappa^2\tau^2 & \text{if} \quad X_2^{ND} \leq \frac{k_2}{\kappa} \end{cases}$$

and

$$I_2^{D,2} = \begin{cases} 2\lambda^2\pi^2\frac{-\lambda^3\pi^3+(\lambda+1)\lambda^2\pi^2+2(1-\lambda)\lambda\pi+(\lambda-1)^2}{\left((\lambda\pi+1-\lambda)^2+\lambda^2\pi^2\right)(\lambda\pi+1-\lambda)}\kappa^2\tau^2 & \text{in the case} \quad U_3^{\mathcal{E}_2^D} + I_1 \\ \frac{(1+\lambda-\pi\lambda)}{(\lambda\pi+1-\lambda)}\lambda^2\pi^2\kappa^2\tau^2 & \text{in the case} \quad X_2^{ND} \leq \frac{k_2}{\kappa} \end{cases}$$

Direct computations yield $I_2^{D,2}\left(U_3^{\mathcal{E}_2^D} + I_1\right) > I_2^{ND,2}($ when $X_2^{ND} > \frac{k_2}{\kappa})$, while $I_2^{D,2}\left(U_4^{\mathcal{E}_2^D} + I_3\right) < I_2^{ND,2}($ when $X_2^{ND} > \frac{k_2}{\kappa})$. ■

Proof or Proposition 4. See Internet Appendix III. ■

Proof of Proposition 5. See Internet Appendix III. ■

Proof of Proposition 6. See Internet Appendix III. ■

Proposition 8 *If $k_1 = 1$, then $\kappa_{MO-LO}^I\tau < \sigma$ and $PIN \geq \psi_{LO-NT}^U$, which implies that the optimal strategy profile at $t = 1$ is*

$$\begin{cases} (BMO, SMO, NT, NT) & \textit{if} \quad \theta_1^I \leq \overline{\theta}_{MO-DO} \\ (BDO, SDO, NT, NT) & \textit{if} \quad \widehat{\theta}_{DO-NT}^3 < \theta_1^I \end{cases}$$

where

$$\widehat{\theta}_{DO-NT}^3 = \begin{cases} \overline{\theta}_{MO-DO} & \textit{if} \quad \theta_2^I \leq \underline{\theta} \\ \widetilde{\theta}_{LO-DO} & \textit{if} \quad \theta_2^I > \underline{\theta}. \end{cases} \tag{B.16}$$

The beliefs of an uninformed trader at $t = 2$, the uninformed and informed traders optimal strategies are characterized in Tables B.8 and a subset of Table B.5 for the equilibrium for which the optimal strategy at $t = 1$ is (BMO, SMO, NT, NT), and in Tables B.12 and a subset of Table B.5, for the equilibrium for which the optimal strategy at $t = 1$ is (BDO, SDO, NT, NT), respectively.

Proof of Proposition 8. Substituting $k_1 = 1$ into the expressions of κ_{MO-LO}^I and ψ_{LO-NT}^U, we have that

$$\begin{aligned} \kappa_{MO-LO}^I &= \frac{1}{1-\frac{1}{2}\delta(1-\lambda)}, \text{ and} \\ \psi_{LO-NT}^U &= 0. \end{aligned}$$

Moreover, since $\kappa_{MO-LO}^I < 2$, it follows that $\kappa_{MO-LO}^I\tau < \sigma$ and $PIN \geq \psi_{LO-NT}^U$ always. In this case, when there is access to the dark pool out of the 6 equilibria there are only two possible equilibria that satisfy these conditions: $\mathcal{E}_2^D$ and $\mathcal{E}_6^D$. From Lemma 4 we can see that $\mathcal{E}_2^{ND}$ is an equilibrium if conditions $(B.7)$ are satisfied, while $\mathcal{E}_6^D$ is an equilibrium if

$$\theta_1^I > \max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} \text{ and } \theta_2^I \leq \underline{\theta},$$

or

$$\widetilde{\theta}_{LO-DO} < \theta_1^I, \text{ and } \underline{\theta} < \theta_2^I.$$

Given that $\kappa^I_{MO-LO}\tau < \sigma$, it follows that informed prefer MO to LO and, therefore, $\overline{\theta}_{LO-DO} < \overline{\theta}_{MO-DO}$, which implies $\max\{\overline{\theta}_{MO-DO}, \overline{\theta}_{LO-DO}\} = \overline{\theta}_{MO-DO}$. Hence, the previous conditions can be rewritten as

$$\theta_1^I > \overline{\theta}_{MO-DO} \text{ and } \theta_2^I \leq \underline{\theta},$$

or

$$\widetilde{\theta}_{LO-DO} < \theta_1^I \text{ and } \underline{\theta} < \theta_2^I.$$

As a result, the optimal strategy profiles of a trader at $t = 1$ are

$$\begin{cases} (BMO, SMO, NT, NT) & \text{if} \quad \theta_1^I \leq \overline{\theta}_{MO-DO} \\ (BDO, SDO, NT, NT) & \text{if} \quad \widehat{\theta}^3_{DO-NT} < \theta_1^I \end{cases}$$

where the expression of $\widehat{\theta}^3_{DO-NT}$ is given in Equation (B.16). ■